

Degenerate billiards in celestial mechanics

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Abstract

In an ordinary billiard trajectories of a Hamiltonian system are elastically reflected after a collision with a hypersurface (scatterer). If the scatterer is a submanifold of codimension more than one, we say that the billiard is degenerate. Degenerate billiards appear as limits of systems with singularities in celestial mechanics. We prove the existence of trajectories of such systems shadowing trajectories of the corresponding degenerate billiards. This research is motivated by the problem of second species solutions of Poincaré.

1 Introduction

1.1 Degenerate billiards

Consider a Hamiltonian system (M, H) with the configuration space M and a classical smooth¹ Hamiltonian H on the phase space T^*M :

$$H(q, p) = \frac{1}{2}\|p - w(q)\|^2 + W(q), \quad (1.1)$$

Here $\|\cdot\|$ is a Riemannian metric on M , and w a covector field representing gyroscopic (or magnetic) forces. The symplectic structure $dp \wedge dq$ on T^*M is standard, so we do not include it in the notation. Let²

$$L(q, \dot{q}) = \max_p (\langle p, \dot{q} \rangle - H(q, p)) = \frac{1}{2}\|\dot{q}\|^2 + \langle w(q), \dot{q} \rangle - W(q) \quad (1.2)$$

be the corresponding Lagrangian.

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¹ C^4 is enough. We do not attempt to lower regularity since in applications to celestial mechanics H is real analytic.

²We use the same notation $\|\cdot\|$ for the norm of a vector and a covector.

Remark 1.1. A transformation $p \rightarrow p + w(q)$ replaces H with a natural Hamiltonian

$$H(q, p) = \frac{1}{2} \|p\|^2 + W(q). \quad (1.3)$$

Then the symplectic structure is twisted

$$\omega = dp \wedge dq + \pi^* \Omega, \quad \pi : T^*M \rightarrow M, \quad (1.4)$$

where $\Omega = dw$ is the gyroscopic 2-form³ on M .

Conversely, if the 2-form Ω is exact, we can make the symplectic structure (1.4) standard and the Hamiltonian takes the form (1.1). The twisted symplectic structure is convenient for many purposes, i.e. reduction of symmetry. However for simplicity we will use the standard form $dp \wedge dq$ and the Hamiltonian (1.1).

Let $N \subset M$ be a submanifold in M which is called a scatterer. Suppose that when a trajectory⁴ $q(t)$ meets the scatterer at a collision point $x = q(\tau) \in N$, it is reflected according to the elastic reflection law⁵

$$\Delta p(\tau) = p_- - p_+ \perp T_x N, \quad p_{\pm} = p(\tau \mp 0), \quad (1.5)$$

$$\Delta H(\tau) = H(x, p_-) - H(x, p_+) = 0. \quad (1.6)$$

Thus the tangent component $y \in T_x^* N$ of the momentum $p \in T_x^* M$ and the energy $H = E$ are preserved. We always assume that the momentum has a jump at the collision: $\Delta p(\tau) \neq 0$. Then also collision velocities $v_{\pm} = \dot{q}(\tau \mp 0)$ have a jump $\Delta v(\tau) = v_- - v_+ \neq 0$ orthogonal to N with respect to the Riemannian metric and they are not tangent to the scatterer: $v_{\pm} \notin T_x N$. By conservation of energy

$$\|v_+\|^2 = \|v_-\|^2 = 2(E - W(x)). \quad (1.7)$$

When N is a hypersurface bounding a domain Ω in M , we obtain a usual billiard system (Ω, N, H) . If

$$d = \text{codim } N > 1,$$

we say that (M, N, H) is a degenerate billiard. We do not assume N to be connected, it may have connected components of different dimension.

Trajectories of the degenerate billiard having collisions with N form a zero measure set in the phase space. Moreover p_+ does not determine p_- uniquely by (1.5)–(1.6): for given p_+ the set of possible p_- has dimension $d - 1$. Thus the past of a collision trajectory does not determine its future. The simplest case is when N is a discrete set in M , then only condition (1.7) remains and a trajectory can be reflected in any direction.

³The covector field w is regarded as a 1-form on M .

⁴A trajectory $q(t)$ is the projection of a solution $(q(t), p(t))$ to the configuration space.

⁵Here p_- is the momentum after the collision, and p_+ before the collision. The strange notation is chosen to fit with the notation for the initial and final momenta of a collision orbit $\gamma : [t_-, t_+] \rightarrow M$ in (2.4).

We are interested in trajectories $\gamma : [\alpha, \beta] \rightarrow M$ with multiple collisions which are called collision chains. They are extremals of the action functional

$$I(\gamma) = \int_{\alpha}^{\beta} L(\gamma(t), \dot{\gamma}(t)) dt = \sum_{j=0}^n I(\gamma_j), \quad \gamma_j = \gamma|_{[t_j, t_{j+1}]}, \quad (1.8)$$

on the set of curves $\gamma : [\alpha, \beta] \rightarrow M$ with fixed end points $a = \gamma(\alpha)$ and $b = \gamma(\beta)$, subject to the constraints $\gamma(t_j) = x_j \in N$, $j = 1, \dots, n$, for some sequence $\alpha = t_0 < t_1 < \dots < t_n < t_{n+1} = \beta$. Here x_j, t_j are independent variables. Each segment $\gamma_j = \gamma|_{[t_j, t_{j+1}]}$ is a collision orbit joining points in N and

$$\Delta p(t_j) \perp T_{x_j} N, \quad \Delta H(t_j) = 0. \quad (1.9)$$

We also require the jump condition

$$\Delta p(t_j) \neq 0. \quad (1.10)$$

An infinite collision chain $\gamma : \mathbb{R} \rightarrow M$ is a concatenation of a sequence $\gamma = (\gamma_j)_{j \in \mathbb{Z}}$ of collision orbits $\gamma_j : [t_j, t_{j+1}] \rightarrow M$ such that the elastic reflection law (1.9)–(1.10) is satisfied at each collision.

An evident source of degenerate billiards are billiards with thin scatterers. Let N be a submanifold in M and N_{ε} its tubular ε -neighborhood with the boundary $\Sigma_{\varepsilon} = \partial N_{\varepsilon}$. Consider the billiard system $(\Omega_{\varepsilon}, \Sigma_{\varepsilon}, H)$ in the domain $\Omega_{\varepsilon} = M \setminus N_{\varepsilon}$ with the boundary $\partial \Omega_{\varepsilon} = \Sigma_{\varepsilon}$ and Hamiltonian H . As $\varepsilon \rightarrow 0$, it approaches the degenerate billiard (M, N, H) with the scatterer N . In [7] it is proved that for small $\varepsilon > 0$ nondegenerate collision chains of this degenerate billiard are shadowed by trajectories of the billiard system in Ω_{ε} . For a discrete set N , this was shown earlier in [14], see also [12].

The goal of the present paper is to show how degenerate billiards appear in Hamiltonian systems with Newtonian singularities. The motivation is the study of periodic and chaotic second species solutions of Poincaré in celestial mechanics [26], see section 1.3. It turns out that the problem is reduced to understanding the corresponding degenerate billiard.

The results of this paper generalize some results of [8] where N was a discrete set and of [4, 10] where N was 2-dimensional.

1.2 Systems with Newtonian singularities

Consider a Hamiltonian system $(M \setminus N, H_{\mu})$ on $T^*(M \setminus N)$ with a classical smooth⁶ Hamiltonian

$$H_{\mu}(q, p) = \frac{1}{2} \|p - w_{\mu}(q)\|_{\mu}^2 + W_{\mu}(q) + \mu V(q, \mu) \quad (1.11)$$

depending on a small parameter $\mu \in (-\mu_0, \mu_0)$. Here $\|\cdot\|_{\mu}$ is a Riemannian metric on M , smoothly depending on μ , and w_{μ} and W_{μ} are covector field and a

⁶ C^4 is enough.

function on M smoothly depending on μ . The potential V is smooth on $M \setminus N$ but undefined on N .

We say that V has a Newtonian singularity on N if in a tubular neighborhood of N there exists a smooth positive function ϕ such that

$$V(q, \mu) = -\frac{\phi(q, \mu)}{d_\mu(q, N)}. \quad (1.12)$$

The distance d_μ is defined by the Riemannian metric $\|\cdot\|_\mu$. If $\mu < 0$, the singular force is repelling (like the Coulomb force), and if $\mu > 0$ attracting (like the gravitational force).

For $\mu = 0$ the singularity disappears and we obtain the Hamiltonian system (M, H_0) with Hamiltonian $H_0 = H$ as in (1.1). The perturbation consists of two parts: regular perturbation which is a smooth function on T^*M , and a singular part μV .

We are interested in nearly collision trajectories of system $(M \setminus N, H_\mu)$ which pass $O(\mu)$ -close to N . Their limits as $\mu \rightarrow 0$ are collision chains of the degenerate billiard (M, N, H_0) with Hamiltonian H_0 and scatterer N . We will give precise statements in section 2.

Remark 1.2. For $\mu < 0$ (repelling force) trajectories of system $(M \setminus N, H_\mu)$ do not have collisions with N . For $\mu > 0$ collisions may appear. However, the Hamiltonian flow on the energy level $\{H_\mu = E\}$ is regularizable (see section 4), since collisions with N are of the type of double collisions in celestial mechanics. After a change of variables and a time reparametrization we obtain a smooth flow without singularities.

1.3 Examples

1. The n center problem. Suppose a particle moves in \mathbb{R}^3 under the gravitational forces of n fixed centers a_1, \dots, a_n with small masses $m_i = \mu\alpha_i$, $0 < \mu \ll 1$. By a time change $t \rightarrow t/\sqrt{\mu}$ this is equivalent to the case of centers of finite masses α_i and large energy of order μ^{-1} of the particle. Then

$$H_\mu(q, p) = \frac{1}{2}|p|^2 + \mu V(q), \quad V(q) = -\sum_{i=1}^n \frac{\alpha_i}{|q - a_i|}, \quad q \in \mathbb{R}^3. \quad (1.13)$$

The limit system is the degenerate billiard $(\mathbb{R}^3, \{a_1, \dots, a_n\}, H_0)$ with a finite scatterer and Hamiltonian $H_0 = |p|^2/2$. Collision chains are polygons with vertices a_i . If $n \geq 4$ there is a Cantor set of collision chains, see [21]. In fact n center problem in \mathbb{R}^3 has chaotic invariant sets on positive energy levels for $n \geq 3$ and any $\mu > 0$ for purely topological reasons, see [9].

2. A more realistic example is the restricted $n + 2$ body problem. Then the bodies a_1, \dots, a_n with small masses $m_i = \mu\alpha_i$ move around the Sun with mass $1 - \mu$ along circular orbits with the same angular velocity $\omega \in \mathbb{R}^3$. An Asteroid of negligible mass moves under the action of the gravitational forces of the Sun

and the small bodies. Then in a rotating coordinate frame,

$$H_\mu(q, p) = \frac{1}{2}|p - \omega \times q|^2 - \frac{1}{|q|} + \mu V(q) + O(\mu),$$

where $V(q)$ is as in (1.13). The corresponding degenerate billiard $(\mathbb{R}^3 \setminus \{0\}, \{a_1, \dots, a_n\}, H_0)$ has the same scatterer as in example 1, but now H_0 is the Hamiltonian of the Kepler problem in a rotating coordinate frame. Because of this the set of collision chains with fixed energy $H_0 = E$ (called Jacobi integral) is very rich: already for $n = 1$ it is a Cantor set, and there is a hyperbolic chaotic set of shadowing orbits, see [8]. Shadowing periodic orbits are called second species solutions of Poincaré. They are well studied for the circular restricted 3 body problem, see e.g. [19, 25, 24, 8, 18] and for the elliptic restricted 3 body problem, see e.g. [19, 6, 4]. Poincaré [26] considered the nonrestricted 3 body problem, see example 4 below.

3. The n body problem with small masses $m_i = \mu\alpha_i$ (or finite masses and large energy). Then after a time change,

$$H_\mu(q, p) = \frac{1}{2}\|p\|^2 + \mu V(q), \quad q = (q_1, \dots, q_n) \in \mathbb{R}^{3n},$$

where

$$\|p\|^2 = \sum_{i=1}^n \frac{|p_i|^2}{\alpha_i}, \quad V(q) = \sum_{i \neq j} \frac{\alpha_i \alpha_j}{|q_i - q_j|}.$$

The limit system is the degenerate billiard $(\mathbb{R}^{3n}, \Delta, H_0)$ with the scatterer

$$\Delta = \cup_{i \neq j} \{q \in \mathbb{R}^{3n} : q_i = q_j\}$$

and Hamiltonian $H_0 = \|p\|^2/2$. The scatterer is not a manifold, so to obtain a billiard of the type studied in this paper we need to exclude from Δ multiple collisions. Dynamics of this billiard is finite: after a bounded number of collisions the bodies escape to infinity. This is a deep result proved in [13], see also [17]. Hence this degenerate billiard does not have invariant sets, in particular it has no periodic orbits and no chaotic hyperbolic sets which are of interest to us.

4. The most important example was introduced by Poincaré [26]. Consider the $n + 1$ body problem with one of the masses m_0 much larger than the rest. Set

$$\frac{m_i}{m_0} = \mu\alpha_i, \quad \sum_{i=1}^n \alpha_i = 1, \quad \mu \ll 1.$$

We may assume that the center of mass is at rest: $\sum_{i=0}^n p_i = 0$. Let q_i be the relative position of m_i with respect to m_0 . Then after a time change we obtain the Hamiltonian

$$H_\mu(q, p) = H_0(q, p) + \frac{\mu}{2} \left| \sum_{i=1}^n p_i \right|^2 - \mu \sum_{i \neq j} \frac{\alpha_i \alpha_j}{|q_i - q_j|}, \quad (1.14)$$

where $q = (q_1, \dots, q_n) \in \mathbb{R}^{3n}$ and

$$H_0 = \sum_{i=1}^n \left(\frac{|p_i|^2}{2\alpha_i} - \frac{\alpha_i}{|q_i|} \right). \quad (1.15)$$

The Hamiltonian H_0 describes n uncoupled Kepler problems which are of course integrable. However for $n \geq 2$ the corresponding degenerate billiard $((\mathbb{R}^3 \setminus \{0\})^n, \Delta, H_0)$ has complicated chaotic dynamics. Orbits of the $n + 1$ body problem shadowing collision chains of this billiard are called second species solutions of Poincaré [26]. Poincaré discussed such solutions for the 3 body problem, but did not provide a rigorous proof of their existence. There are many works of Astronomers on the subject but few mathematical results (except for the restricted circular 3 body problem, see e.g. [25, 19, 24, 8] and the elliptic restricted 3 body problem, see e.g. [19, 6, 4]). Some rigorous results for the unrestricted plane 3 body problem were proved in [10, 11] and for the 2 center - 2 body problem in [16].

2 Shadowing collision chains

2.1 Discrete Lagrangian system of a degenerate billiard

Before formulating the main results we need to recall some definitions from [7], see also [10].

The Hamiltonian is constant along collision chains of a degenerate billiard, so let us fix energy $H = E$. The restriction of the Hamiltonian system (M, H) to the energy level will be denoted $(M, H = E)$. Trajectories $\gamma : [\alpha, \beta] \rightarrow M$ with energy E are extremals of the Maupertuis action $J = J_E$:

$$J(\gamma) = \int_{\alpha}^{\beta} g_E(\gamma(t), \dot{\gamma}(t)) dt, \quad (2.1)$$

i.e. geodesics⁷ of the Jacobi metric [1, 2]

$$g_E(q, \dot{q}) = \max_p \{ \langle p, \dot{q} \rangle : H(q, p) = E \} = \sqrt{2(E - W(q))} \|\dot{q}\| + \langle w(q), \dot{q} \rangle \quad (2.2)$$

in the domain of possible motion

$$\mathcal{D}_E = \{q \in M : W(q) < E\}. \quad (2.3)$$

Remark 2.1. *The metric g_E is positive definite in the domain*

$$\{q \in M : W(q) + \|w(q)\|^2/2 < E\},$$

but not in \mathcal{D}_E , so g_E is not a Finsler metric in \mathcal{D}_E . However g_E is convex in the velocity, so local calculus of variations works. In particular, for any $x_0 \in \mathcal{D}_E$ there is $r > 0$ such that a pair of points in the ball $B_r(x_0)$ is joined by a geodesic in $B_r(x_0)$.

⁷We identify curves which differ by an orientation preserving reparametrization.

For trajectories γ with energy E ,

$$J(\gamma) = \int_{\gamma} p dq.$$

When the energy is fixed, we denote the degenerate billiard by $(M, N, H = E)$. As in section 1, we call a trajectory $\gamma : [t_-, t_+] \rightarrow M$ a collision orbit if its end points lie in N and there is no tangency and no early collisions with the scatterer:

$$\gamma(t_{\pm}) = a_{\pm} \in N, \quad v_{\pm} = \dot{\gamma}(t_{\pm}) \notin T_{a_{\pm}}N, \quad \gamma(t) \notin N, \quad t_- < t < t_+. \quad (2.4)$$

In particular, $a_{\pm} \in \mathcal{D}_E \cap N$.

We call γ nondegenerate if it is nondegenerate as a critical point of J , i.e. the points a_- and a_+ are non-conjugate. Then there exist neighborhoods $U_{\pm} \subset M$ of a_{\pm} such that for all $q_{\pm} \in U_{\pm}$ there exists an orbit $\gamma(q_-, q_+)$ with energy E joining q_- and q_+ , and it smoothly depends on q_-, q_+ . The Maupertuis action

$$S(q_-, q_+) = J(\gamma(q_-, q_+)) \quad (2.5)$$

is a smooth function on $U_- \times U_+$. The initial and final momenta of the orbit γ are

$$p_- = -D_{q_-} S, \quad p_+ = D_{q_+} S.$$

The twist of the action function is the linear transformation

$$B(q_-, q_+) = D_{q_-} D_{q_+} S : T_{q_-} M \rightarrow T_{q_+}^* M,$$

i.e. a bilinear form on $T_{q_-} M \times T_{q_+} M$. Since the Hamiltonian system is autonomous, it is always degenerate:

$$B(a_-, a_+)v_- = 0, \quad B^*(a_-, a_+)v_+ = 0. \quad (2.6)$$

We say that the collision orbit γ has nondegenerate twist if the restriction of the bilinear form $B(a_-, a_+)$ to $T_{a_-}N \times T_{a_+}N$ is nondegenerate. For this it is necessary that $v_{\pm} \notin T_{a_{\pm}}N$, i.e. the collision orbit is not tangent to the scatterer N at the end points. For an ordinary billiard, when N is a hypersurface, this is also sufficient for the nondegenerate twist, but in general not for a degenerate billiard.

If γ has nondegenerate twist, the restriction of S to a neighborhood of (a_-, a_+) in $N \times N$ is the generating function of a locally defined symplectic map $f : V^- \rightarrow V^+$ of open sets $V^{\pm} \subset T^*N$:

$$f(x_-, y_-) = (x_+, y_+) \Leftrightarrow y_+ = D_{x_+} S, \quad y_- = -D_{x_-} S.$$

Here $y_{\pm} = p_{\pm}|_{T_{x_{\pm}}N} \in T_{x_{\pm}}^*N$ are the tangent projections of the collision momenta. Hence $V^{\pm} \subset \mathcal{M}_E$, where

$$\mathcal{M}_E = \{(x, y) \in T^*N : F(x, y) < E\}, \quad (2.7)$$

$$F(x, y) = \min_{p|_{T_x N = y}} H(x, p) = \frac{1}{2} \|y - a(x)\|^2 + W(x). \quad (2.8)$$

Here $a(x) = w(x)|_{T_x N} \in T_x^* N$. The Riemannian metric is the induced metric on N . Thus F is the Hamiltonian on $T^* N$ corresponding to the Lagrangian $L|_{T N}$.

Remark 2.2. *If the symplectic structure (1.4) is twisted, then, locally, $\Omega = dw$, where w is defined up to adding a differential $d\varphi$. Then the generating function $S(x_-, x_+)$ is defined up to adding a cocycle $\varphi(x_+) - \varphi(x_-)$.*

In general there may exist several (or none) nondegenerate collision orbits with energy E joining a pair of points in N . Thus we obtain a collection $\mathcal{L} = \{L_k\}_{k \in K}$ of action functions (2.5) on open sets $U_k \subset N \times N$. Under the twist condition, L_k generates a local symplectic map $f_k : V_k^- \rightarrow V_k^+$ of open sets in \mathcal{M}_E . We call the partly defined multivalued “map” $\mathcal{F} = \{f_k\}_{k \in K}$ of \mathcal{M}_E the collision map, or the scattering map of the degenerate billiard. It is analogous to the scattering map of a normally hyperbolic invariant manifold, see [15]. The degenerate billiard defines a discrete dynamical system – the skew product of the maps $\mathcal{F} = \{f_k\}_{k \in K}$ which is a map of a subset in $K^{\mathbb{Z}} \times \mathcal{M}_E$.

Remark 2.3. *Computation of the collision map is usually difficult. See e.g. [6, 10] for the degenerate billiards appearing in the elliptic restricted 3 body problem and in the nonrestricted plane 3 body problem.*

An orbit of \mathcal{F} is a pair (\mathbf{k}, \mathbf{z}) of sequences $\mathbf{k} = (k_j)$, $\mathbf{z} = (z_j)$, where $z_j = (x_j, y_j) \in V_{k_j}^- \cap V_{k_{j+1}}^+$, such that $z_{j+1} = f_{k_j}(z_j)$. The orbit (\mathbf{k}, \mathbf{z}) defines a chain of collision orbits γ_j joining x_j with x_{j+1} . The tangent collision momenta of the collision chain are

$$y_j = D_{x_j} L_{k_{j-1}}(x_{j-1}, x_j) = -D_{x_j} L_{k_j}(x_j, x_{j+1}). \quad (2.9)$$

Also without the twist condition, the degenerate billiard $(M, N, H = E)$ can be viewed as a discrete Lagrangian system (DLS) with multivalued Lagrangian $\mathcal{L} = \{L_k\}_{k \in K}$, see [12]. Infinite collision chains correspond to critical points $\mathbf{x} = (x_j)_{j \in \mathbb{Z}}$ of the discrete action functional

$$\mathcal{A}_{\mathbf{k}}(\mathbf{x}) = \sum_{j \in \mathbb{Z}} L_{k_j}(x_j, x_{j+1}), \quad (x_j, x_{j+1}) \in U_{k_j}. \quad (2.10)$$

For infinite collision chains, the sum makes no sense, so $\mathcal{A}_{\mathbf{k}}(\mathbf{x})$ is a formal functional, but the derivative

$$\mathcal{A}'_{\mathbf{k}}(\mathbf{x}) = (D_{x_j} \mathcal{A}_{\mathbf{k}}(\mathbf{x}))_{j \in \mathbb{Z}}, \quad D_{x_j} \mathcal{A}_{\mathbf{k}}(\mathbf{x}) = D_{x_j} (L_{k_{j-1}}(x_{j-1}, x_j) + L_{k_j}(x_j, x_{j+1}))$$

is well defined. A trajectory of the DLS is a pair $(\mathbf{k}, \mathbf{x}) \in K^{\mathbb{Z}} \times N^{\mathbb{Z}}$ such that $\mathcal{A}'_{\mathbf{k}}(\mathbf{x}) = 0$. We call the trajectory (\mathbf{k}, \mathbf{x}) admissible if the corresponding collision chain satisfies the jump condition (1.10).

The Hessian

$$\mathcal{A}''_{\mathbf{k}}(\mathbf{x}) = (D_{x_i} D_{x_j} \mathcal{A}_{\mathbf{k}}(\mathbf{x}))_{i, j \in \mathbb{Z}}$$

of the action functional is 3-diagonal:

$$\mathcal{A}''_{\mathbf{k}}(\mathbf{x}) \mathbf{u} = \mathbf{v}, \quad v_i = B_{i-1} u_{i-1} + A_i u_i + B_i^* u_{i+1}, \quad (2.11)$$

where

$$B_i = D_{x_i} D_{x_{i+1}} L_{k_i}(x_i, x_{i+1})$$

is the twist of the collision orbit γ_i . The variational equation of the trajectory (\mathbf{k}, \mathbf{x}) is $A''_{\mathbf{k}}(\mathbf{x})\mathbf{u} = 0$. Under the twist condition, B_i is invertible, and the variational equation defines the linear Poincaré map $P_i : (u_{i-1}, u_i) \rightarrow (u_i, u_{i+1})$.

For n -periodic collision chains, \mathbf{x} is a critical point of the periodic action functional

$$\mathcal{A}_{\mathbf{k}}^{(n)}(\mathbf{x}) = \sum_{j=0}^{n-1} L_{k_j}(x_j, x_{j+1}), \quad \mathbf{x} = (x_1, \dots, x_n), \quad x_n = x_0. \quad (2.12)$$

We call the periodic collision chain nondegenerate if \mathbf{x} is a nondegenerate critical point of $\mathcal{A}_{\mathbf{k}}^{(n)}$. If the twist condition holds, this is equivalent to the usual nondegeneracy condition $\det(P - I) \neq 0$, where $P = P_n \circ \dots \circ P_1$ is the linear monodromy map.

Finite collision chains joining the points $a, b \in M$ correspond to critical points of a finite sum

$$\mathcal{A}_{\mathbf{k}}^{a,b}(\mathbf{x}) = \sum_{j=0}^n L_{k_j}(x_j, x_{j+1}), \quad x_0 = a, \quad x_{n+1} = b, \quad \mathbf{x} = (x_1, \dots, x_n). \quad (2.13)$$

We call the finite collision chain nondegenerate if the critical point \mathbf{x} is nondegenerate.

Dynamics of the DLS is represented by the translation

$$\mathcal{T} : K^{\mathbb{Z}} \times N^{\mathbb{Z}} \rightarrow K^{\mathbb{Z}} \times N^{\mathbb{Z}}, \quad (k_j, x_j) \rightarrow (k_{j+1}, x_{j+1}).$$

If \mathcal{T} has a compact⁸ invariant set $\Lambda \subset K^{\mathbb{Z}} \times N^{\mathbb{Z}}$ of trajectories of the DLS, and the collision map \mathcal{F} is well defined, then it will have a compact invariant set $\tilde{\Lambda} \subset K^{\mathbb{Z}} \times N^{\mathbb{Z}}$ with $\mathcal{F} : \tilde{\Lambda} \rightarrow \tilde{\Lambda}$ topologically conjugate to $\mathcal{T} : \Lambda \rightarrow \Lambda$.

The usual definition of a hyperbolic set is formulated in terms of the dichotomy of solutions of the variational equation. It works under the twist condition, when the linear Poincaré maps P_i are well defined.

A trajectory (\mathbf{k}, \mathbf{x}) of the DLS is hyperbolic if for any $j \in \mathbb{Z}$ there are stable and unstable subspaces $E_j^{\pm} \subset T_{x_j} N \times T_{x_{j+1}} N$ such that $E_j^+ \cap E_j^- = \{0\}$ and for any solution $\mathbf{u} = (u_i)$ of the variational equation $w_j = (u_j, u_{j+1}) \in E_j^+$ implies $w_i = (u_i, u_{i+1}) \in E_i^+$ for all $i > j$. Moreover w_i decreases exponentially as $i \rightarrow \infty$: there is $C > 0$ and $\lambda \in (0, 1)$ such that

$$\|w_i\| \leq C\lambda^{i-j}\|w_j\|, \quad i > j.$$

Similarly for the unstable subspace: $w_j = (u_j, u_{j+1}) \in E_j^-$ implies $w_i = (u_i, u_{i+1}) \in E_i^-$ for all $i < j$ and w_i decreases exponentially as $i \rightarrow -\infty$:

$$\|w_i\| \leq C\lambda^{j-i}\|w_j\|, \quad i < j.$$

⁸The topology on $K^{\mathbb{Z}} \times N^{\mathbb{Z}}$ is the product topology.

A compact \mathcal{T} -invariant set Λ of trajectories is hyperbolic if this holds for every trajectory $(\mathbf{k}, \mathbf{x}) \in \Lambda$ with C, λ independent of the trajectory.

For our purposes another definition, not requiring the twist condition, is more convenient. If we use the Riemannian metric to identify $T_{x_i}N$ and $T_{x_i}^*N$, the Hessian $\mathcal{A}_{\mathbf{k}}''(\mathbf{x})$ becomes a linear operator $\mathcal{A}_{\mathbf{k}}''(\mathbf{x}) : l_\infty \rightarrow l_\infty$, where l_∞ is the Banach space of sequences

$$\mathbf{u} = (u_i)_{i \in \mathbb{Z}}, \quad u_i \in T_{x_i}N, \quad \|\mathbf{u}\|_\infty = \sup_i \|u_i\| < \infty.$$

If Λ is a compact invariant set of the DLS, then the Hessian is a bounded operator: $\|\mathcal{A}_{\mathbf{k}}''(\mathbf{x})\|_\infty \leq c = c(\Lambda)$ for any $(\mathbf{k}, \mathbf{x}) \in \Lambda$.

Definition 2.1. *We say that the trajectory (\mathbf{k}, \mathbf{x}) is hyperbolic if the Hessian $\mathcal{A}_{\mathbf{k}}''(\mathbf{x})$ has bounded inverse in the l_∞ norm. We say that a compact \mathcal{T} -invariant set $\Lambda \subset K^\mathbb{Z} \times N^\mathbb{Z}$ of trajectories of the DLS is hyperbolic if this is true for all trajectories: $\|\mathcal{A}_{\mathbf{k}}''(\mathbf{x})^{-1}\|_\infty \leq C$ with $C = C(\Lambda)$ independent of the trajectory $(\mathbf{k}, \mathbf{x}) \in \Lambda$.*

If the twist condition holds, then, as shown in [3], this definition of hyperbolicity is equivalent to the standard one.⁹ But Definition 2.1 makes sense also without the twist condition, for example when N has connected components of different dimension, so the twist condition evidently fails.

2.2 Main results

Consider the system $(M \setminus N, H_\mu)$ with Newtonian singularity on N and the corresponding degenerate billiard (M, N, H_0) with Hamiltonian (1.15). Fix energy E .

Theorem 2.1. *Let γ be a nondegenerate periodic collision chain of the degenerate billiard $(M, N, H_0 = E)$. There exists $\mu_0 > 0$ such that for any $\mu \in I_{\mu_0} = (-\mu_0, 0) \cup (0, \mu_0)$ the chain γ is shadowed by a periodic orbit γ_μ of the system $(M \setminus N, H_\mu = E)$.*

The shadowing error is of order $O(\mu \ln |\mu|)$, i.e. $d(\gamma_\mu(t), \gamma) \leq c|\mu \ln |\mu||$. At each near collision, the shadowing orbit γ_μ passes at a distance $\leq c\mu$ from N . However, for $\mu > 0$ (attracting singularity) it may have collisions with N . The regularized flow on the level $\{H_\mu = E\}$ has no singularity, so dynamics is always well defined. If, for physical reasons, we need to avoid regularizable collisions, we have to impose an extra condition on the collision chain $\gamma = (\gamma_j)$. Let

$$v_j^\pm = \dot{\gamma}(t_j \pm 0)$$

be the collision velocities at j -th collision point $x_j = \gamma(t_j)$, and let u_j^\pm be their projections to the quotient space $T_{x_j}M/T_{x_j}N$. The jump condition $\Delta p_j(t_j) \neq 0$

⁹In [3] a single valued discrete Lagrangian was considered, but in general the proof is the same.

implies $u_j^+ \neq u_j^-$. For $\mu > 0$ we assume the no straight reflection condition $u_j^+ \neq -u_j^-$:

$$v_j^+ + v_j^- \notin T_{x_j}N \quad \text{for all } j. \quad (2.14)$$

Then the shadowing trajectory γ_μ will have no collisions: it passes N at the minimal distance

$$c_1\mu \leq d(\gamma_\mu, N) \leq c_2\mu, \quad 0 < c_1 < c_2. \quad (2.15)$$

Condition (2.14) is less essential than the jump condition (1.10) since dynamics is well defined also for trajectories colliding with N . For $\mu < 0$ the no straight reflection condition is not needed.

Remark 2.4. *If the twist condition holds (in particular all components of N have the same codimension d), then the periodic orbit γ_μ has $2d$ large Lyapunov exponents of order $O(\ln|\mu|)$. Thus γ_μ is strongly unstable, even if the corresponding periodic orbit of the DLS is Lyapunov stable.*

Theorem 2.1 is a generalization of a theorem in [10, 11], where it was proved for the case of second species solutions of the plane 3 body problem.

A similar statement holds for collision chains joining given points $a, b \in M \setminus N$.

Theorem 2.2. *Let γ be a nondegenerate collision chain of the degenerate billiard $(M, N, H_0 = E)$ joining the points $a, b \in M \setminus N$. There exists $\mu_0 > 0$ such that for any $\mu \in I_{\mu_0}$ the chain γ is $O(\mu \ln|\mu|)$ -shadowed by an orbit of the system $(M \setminus N, H_\mu = E)$ joining a, b .*

The next theorem gives a hyperbolic invariant set of shadowing trajectories.

Theorem 2.3. *Let $\Lambda \subset K^{\mathbb{Z}} \times N^{\mathbb{Z}}$ be a compact hyperbolic invariant set of the DLS such that all orbits in Λ are admissible. There exists $\mu_0 > 0$ such that for any $\mu \in I_{\mu_0}$ and any orbit $(\mathbf{k}, \mathbf{x}) \in \Lambda$ there exists a trajectory γ_μ of system $(M \setminus N, H_\mu = E)$ shadowing (as a non-parametrized curve) the corresponding collision chain γ of the degenerate billiard $(M, N, H_0 = E)$. Shadowing trajectories form a compact hyperbolic invariant set $\Lambda_\mu \subset \{H_\mu = E\}$ of system $(M \setminus N, H_\mu = E)$.*

The shadowing error is of the same order $O(\mu \ln|\mu|)$ as in Theorem 2.1. Recall that a trajectory (\mathbf{k}, \mathbf{x}) of the DLS is admissible if the corresponding collision chain satisfies the jump condition (1.10). For $\mu > 0$ to avoid collisions we have to assume also the no straight reflections condition (2.14) for trajectories in Λ . Then the shadowing trajectories satisfy (2.15).

Note that in Theorem 2.1 the periodic orbit of the degenerate billiard does not need to be hyperbolic, so Theorems 2.1 and 2.3 are formally independent.

To be honest, one of the main ingredients of the proof of Theorems 2.1–2.3, Theorem 3.2, will be proved only for $d = \text{codim } N \leq 3$. The proof is based on Theorem 4.2 (the generalized Shilnikov lemma) which holds for any codimension. However, to apply Theorem 4.2, we first need to regularize singularities.

We use the Levi-Civita regularization for $d \leq 2$ and KS regularization [23] for $d = 3$. Collisions with N (they are double collisions) are regularizable in any dimension, but standard multidimensional methods of regularization (e.g. Moser's regularization) are less convenient for our purposes since regularization is not well defined in the limit $\mu \rightarrow 0$. However, there is no doubt that Theorem 3.2 is true for any d , just the method of the proof needs to be changed. A multidimensional analog of the KS regularization is the Clifford algebra regularization which should give the proof of Theorem 3.2 for all $d > 3$. We do not consider the case $d > 3$ since it has no applications in celestial mechanics (unless one plans to do celestial mechanics in a space of dimension > 3).

For a discrete scatterer N , Theorems 2.1 and 2.3 were proved in [8] and used to prove the existence of chaotic second species solutions of the restricted circular 3 body problem. A version of these theorems for the elliptic restricted 3 body problem was proved in [4] (then N is one-dimensional). A version of Theorem 2.1 was proved in [10] for the plane nonrestricted 3 body problem. Then N is 2-dimensional but becomes 1-dimensional after reduction of symmetry.

2.3 Shadowing for systems with symmetry

Formally Theorems 2.1 and 2.3 are of little use in celestial mechanics. Indeed, Hamiltonian systems of celestial mechanics usually have translational or rotational symmetry and so they do not possess nondegenerate periodic orbits or hyperbolic invariant sets. Hence Theorems 2.1 and 2.3 do not apply. The exception is Theorem 2.2: it works also in the presence of symmetry. Indeed, symmetry is broken by fixing the end points of a trajectory (if they are not fixed points of the group action), so nondegenerate connecting chains may exist. Restricted problems of celestial mechanics also have symmetry broken and then all Theorems 2.1–2.3 work.

To apply Theorems 2.1 and 2.3 in celestial mechanics, we have to reduce symmetry. We describe the reduction in the simplest situation arising in applications, see also [10]. Suppose the degenerate billiard (M, N, H) has an abelian symmetry group \mathbb{A}^s , where \mathbb{A}^s is a torus $\mathbb{T}^s = \mathbb{R}^s/\mathbb{Z}^s$, or \mathbb{R}^s , or their product (cylinder). More precisely, suppose there is a smooth group action $\Phi_\theta : M \rightarrow M$, $\theta \in \mathbb{A}^s$, which preserves the Hamiltonian and the scatterer:

$$\Phi_\theta(N) = N, \quad H(\Phi_\theta(q), p) = H(q, D\Phi_\theta(q)^*p).$$

For any $\xi \in \mathbb{R}^s$, the one-parameter symmetry group $\Phi_{t\xi}$ is generated by the vector field $u_\xi(q) = X(q)\xi$, where

$$X(q) = D_\theta|_{\theta=0} \Phi_\theta(q) : \mathbb{R}^s \rightarrow T_q M.$$

Let

$$G_\xi(q, p) = \langle u_\xi(q), p \rangle$$

be the corresponding Noether integral [1, 2] of the Hamiltonian system. Then

$$G : T^*M \rightarrow (\mathbb{R}^s)^*, \quad \langle G(q, p), \xi \rangle = G_\xi(q, p), \quad \xi \in \mathbb{R}^s,$$

is the momentum integral. Since u_ξ is tangent to N , G is preserved by the reflection and so it will be also an integral of the degenerate billiard (M, N, H) .

The corresponding DLS with the Lagrangian $\mathcal{L} = \{L_k\}_{k \in K}$ has the symmetry

$$L_k(\Phi_\theta(x_-), \Phi_\theta(x_+)) = L_k(x_-, x_+).$$

The action functional (2.10) is invariant:

$$\mathcal{A}_k(\Phi_\theta \mathbf{x}) = \mathcal{A}_k(\mathbf{x}).$$

Thus for any $\xi \in \mathbb{R}^s$, $\mathbf{u}_\xi = (u_\xi(x_j))_{j \in \mathbb{Z}}$ is in the kernel of the Hessian $\mathcal{A}_k''(\mathbf{x})$, and the Hessian is non-invertible: there are no nondegenerate periodic orbits or hyperbolic trajectories except fixed points of the group Φ_θ .

We call an n -periodic collision chain $\gamma = (\gamma_i)_{i \in \mathbb{Z}}$ nondegenerate modulo symmetry if it has only degeneracy coming from symmetry. The corresponding critical point \mathbf{x} of the action functional (2.12) satisfies

$$D^2 \mathcal{A}_k^{(n)}(\mathbf{x}) \mathbf{v} = 0 \quad \Rightarrow \quad v_i = u_\xi(x_i), \quad \xi \in \mathbb{R}^s.$$

Suppose now that the system $(M \setminus N, H_\mu)$ with Newtonian singularities has a symmetry group:

$$H_\mu(\Phi_\theta(q), p) = H_\mu(q, D\Phi_\theta(q)^* p), \quad \theta \in \mathbb{A}^s.$$

Then Φ_θ is a symmetry group of the corresponding degenerate billiard (M, N, H_0) . We have the following version of Theorem 2.1 for systems with symmetry.

Theorem 2.4. *Let γ be a nondegenerate modulo symmetry periodic collision chain of the degenerate billiard $(M, N, H_0 = E)$. There exists $\mu_0 > 0$ such that for any $\mu \in I_{\mu_0}$ the chain γ is shadowed by a periodic orbit γ_μ of the system $(M \setminus N, H_\mu = E)$.*

Of course γ_μ is defined modulo symmetry $\gamma_\mu \rightarrow \Phi_\theta \gamma_\mu$. In Theorem 2.4 it is not possible to prescribe the value of the momentum integral G of the periodic orbit γ_μ . To find trajectories with given value of G , we need to consider orbits periodic modulo symmetry: $\gamma(t + T) = \Phi_\theta \gamma(t)$.

The discrete action functional (2.12) is modified as follows:

$$\mathcal{P}_k(\mathbf{x}, \theta) = \mathcal{A}_k^{(n)}(\mathbf{x}) - \langle G, \theta \rangle, \quad \mathbf{x} = (x_1, \dots, x_n), \quad \theta \in \mathbb{R}^s, \quad x_n = \Phi_\theta(x_0).$$

Critical points (\mathbf{x}, θ) of \mathcal{P}_k correspond to collision chains γ which are periodic modulo symmetry and have integral G . We call γ nondegenerate if (\mathbf{x}, θ) is a nondegenerate (modulo symmetry) critical point of \mathcal{P}_k .

Theorem 2.5. *Let γ be a nondegenerate periodic modulo symmetry collision chain of the degenerate billiard $(M, N, H = E)$. Let G be its momentum integral. There exists $\mu_0 > 0$ such that for any $\mu \in I_{\mu_0}$ the chain γ is shadowed modulo symmetry by a periodic modulo symmetry orbit γ_μ of the system $(M \setminus N, H_\mu = E)$ with the momentum integral G .*

Shadowing modulo symmetry means that $d(\Phi_{\theta(t)}\gamma_\mu(t), \gamma) \leq c|\mu \ln |\mu||$ for some $\theta(t) \in \mathbb{A}^s$.

To prove Theorem 2.5, we perform symmetry reduction. Suppose that the quotient space $\tilde{M} = M/\Phi_\theta$ is a smooth manifold and the projection $\pi : M \rightarrow \tilde{M}$ is a smooth fiber bundle with fiber \mathbb{A}^s . For simplicity assume that the fibre bundle $\pi : M \rightarrow \tilde{M}$ is trivial. This is always true locally. Then \tilde{M} can be realized as a cross section $\tilde{M} \subset M$ of the group action Φ_θ .

Let L be the Lagrangian (1.2). Define the reduced Lagrangian (Routh function) on $T\tilde{M}$ by

$$\tilde{L}(q, \dot{q}) = \text{Crit}_\xi(L(q, \dot{q} + u_\xi(q)) - \langle G, \xi \rangle), \quad q \in \tilde{M}, \quad \dot{q} \in T_q\tilde{M}, \quad (2.16)$$

where Crit_ξ means taking a critical value with respect to $\xi \in \mathbb{R}^s$. Since L is convex in the velocity, the Routh function is well defined. For the standard definition see [1, 2].

Let \tilde{H} be the Hamiltonian corresponding to \tilde{L} . Then trajectories of the Hamiltonian system (M, H) with the momentum G are projected to trajectories of the reduced Hamiltonian system (\tilde{M}, \tilde{H}) .

If (M, N, H) is a degenerate billiard with symmetry, then the reduced degenerate billiard is $(\tilde{M}, \tilde{N}, \tilde{H})$, where $\tilde{N} = N/\Phi_\theta$.

If the system with singularities $(M \setminus N, H_\mu)$ has a symmetry Φ_θ , then for fixed momentum G , the degenerate billiard corresponding to the reduced system $(\tilde{M} \setminus \tilde{N}, \tilde{H}_\mu)$ will be the reduced billiard $(\tilde{M}, \tilde{N}, \tilde{H}_0)$. Now we can apply Theorem 2.1 to the reduced system with singularities and to the corresponding reduced billiard. This proves Theorem 2.5.

Remark 2.5. *If the group \mathbb{A}^s is a torus or a cylinder, then the fibration is nontrivial in general. Then the construction of the reduced system is local. The global version needs choosing a connection for the fibre bundle $\pi : M \rightarrow \tilde{M}$ and using the symplectic structure twisted by the curvature form of the connection [2]. However this is not needed in this paper since our results are essentially local: we can assume that the collision chains lie in a domain $U \subset M$ such that fibre bundle $\pi : U \rightarrow \tilde{U}$ is trivial.*

We can perform symmetry reduction also for the DLS describing the billiard. For any trajectory (\mathbf{k}, \mathbf{x}) , the Noether integral corresponding to $\xi \in \mathbb{R}^s$ is

$$G_\xi = \langle u_\xi(x_j), y_j \rangle,$$

where y_j is the momentum (2.9).

The fibration $\pi : M \rightarrow \tilde{M}$ defines a fibration $\pi : N \rightarrow \tilde{N}$ to the orbits of the group action. We assume that it is trivial. Then $\tilde{N} = N/\Phi_\theta$ can be identified with a cross section $\tilde{N} \subset N$ of the group action $\Phi_\theta|_N$.

For a fixed value of the integral G , define the reduced discrete Lagrangian (discrete Routh function) by the Legendre transform

$$\tilde{L}_k(x_-, x_+) = \text{Crit}_\theta(L_k(x_-, \Phi_\theta(x_+)) - \langle G, \theta \rangle), \quad x_\pm \in \tilde{N}. \quad (2.17)$$

This requires a twist condition: the bilinear form

$$\xi, \eta \rightarrow \langle B_k(x_-, x_+) u_\xi(x_-), u_\eta(x_+) \rangle, \quad \xi, \eta \in \mathbb{R}^s, \quad (2.18)$$

is nondegenerate. Here $B_k(x_-, x_+) = D_{x_-} D_{x_+} L_k(x_-, x_+)$ is the twist of the Lagrangian L_k . In general the reduced discrete Lagrangian is locally defined: it is a function on an open set $\tilde{U}_k \subset \tilde{N} \times \tilde{N}$.

For any trajectory (\mathbf{k}, \mathbf{x}) of the DLS with momentum integral G setting $\tilde{\mathbf{x}} = (\tilde{x}_j)$, $\tilde{x}_j = \pi(x_j)$, we obtain a trajectory $(\mathbf{k}, \tilde{\mathbf{x}})$ of the reduced DLS with the Lagrangian $\tilde{\mathcal{L}} = \{\tilde{L}_k\}_{k \in K}$. Conversely, a trajectory $(\mathbf{k}, \tilde{\mathbf{x}})$ of the reduced DLS defines a (nonunique) trajectory (\mathbf{k}, \mathbf{x}) of the original DLS with momentum G .

Now we can apply Theorem 2.3 to the reduced Hamiltonian system $(\tilde{M} \setminus \tilde{N}, \tilde{H}_\mu)$ and the corresponding reduced DLS and obtain the existence of hyperbolic modulo symmetry invariant sets on a level set of G for the system $(M \setminus N, H_\mu = E)$ when the corresponding reduced DLS has a compact hyperbolic invariant set.

In the next publication these results will be used to study chaotic second species solutions of the nonrestricted 3 body problem.

3 Proofs

In this section we prove Theorems 2.1 and 2.3. The proof of Theorem 2.2 is similar. The proofs are based on a local connection result – Theorem 3.1 which is proved in section 4.

3.1 Local connection

Let d be the distance in M defined by the Riemannian metric $\|\cdot\| = \|\cdot\|_0$. We parameterize a tubular neighborhood

$$N_\rho = \{q \in M : d(q, N) \leq \rho\}$$

by the exponential map

$$f : T^\perp N \rightarrow M, \quad q = f(x, u) = \exp_x u, \quad x \in N, u \in T_x^\perp N.$$

Let $D \Subset N$ be an open set with compact closure. Then for small $\rho > 0$,

$$U_\rho = \{q = f(x, u) : x \in D, \|u\| \leq \rho\} \subset N_\rho$$

has smooth boundary

$$\Sigma_\rho = \{q = f(x, u) : x \in D, \|u\| = \rho\} \quad (3.1)$$

and f is a diffeomorphism onto U_ρ . For $q \in U_\rho$ we have $d(q, N) = \|u\|$.

Consider a degenerate billiard $(M, N, H_0 = E)$. Suppose that $D \Subset N \cap \mathcal{D}_E$ is contained in the domain of possible motion (2.3). There exists $r > 0$ such

that for any $x_0 \in D$ we have $B_r(x_0) \subset \mathcal{D}_E$ and for any pair of points $q_- \neq q_+$ in the ball $B_r(x_0)$ there exists a trajectory γ of system $(M, H = E)$ (geodesic of the Jacobi metric) joining q_\pm in $B_r(x_0)$. The trajectory γ smoothly depends on $q_+ \neq q_-$. Let $S(q_-, q_+) = J_0(\gamma)$ be its Maupertuis action (2.5).

Fix arbitrary large¹⁰ $C > 0$ and let

$$P_\rho = \{(q_+, q_-) \in \Sigma_\rho^2 : d(q_+, q_-) \leq C\rho\}. \quad (3.2)$$

We will connect a pair of points $q_+, q_- \in P_\rho$ by a billiard trajectory of energy E having a single reflection from N at a point x_0 .

Proposition 3.1. *Let $\rho = \rho(C, D) > 0$ be sufficiently small. Then for any $(q_-, q_+) \in P_\rho$:*

- *There exists $x_0 \in N$ and a trajectory γ of the degenerate billiard $(M, N, H = E)$ joining q_+, q_- in $B_r(x_0) \cap N_\rho$ after a reflection at x_0 . Thus $\gamma = \gamma_+ \cdot \gamma_-$ is a concatenation of a trajectory γ_+ which joins q_+ with x_0 and γ_- which joins x_0 with q_- .*
- *$x_0 = \xi(q_+, q_-)$ and γ smoothly depend on $(q_+, q_-) \in P_\rho$.*
- *The Maupertuis action $J_0(\gamma) = R_0(q_+, q_-)$ is a smooth function on P_ρ and*

$$R_0(q_+, q_-) = \text{Crit}_{x \in N} R(q_+, x, q_-), \quad R = S(q_+, x) + S(x, q_-). \quad (3.3)$$

More precisely, x_0 is the only critical point of $x \rightarrow R(q_+, x, q_-)$ in $N \cap B_r(x_0)$, and it is nondegenerate.

Let $p_\pm \in T_{q_\pm}^* \Sigma_\rho$ be the momenta of γ at $q_\pm \in \Sigma_\rho$. Then $R(q_+, x, q_-)$ is the generating function of the Lagrangian relation \mathcal{R} between the points (q_+, p_+) and (q_-, p_-) . Note that \mathcal{R} is not a map unless N is a hypersurface (ordinary billiard), then $\mathcal{R} : (q_+, p_+) \rightarrow (q_-, p_-)$ is a symplectic map of a set in $T^* \Sigma_\rho$.

Proposition 3.1 is a familiar property of systems with elastic reflections (a version of Fermat' principle). However we give a proof since the notations will be needed in the next theorem.

Since Proposition 3.1 is local: all trajectories lie in a neighborhood of some point $x_0 \in N$, without loss of generality we may assume that $D \subset N$ is contractible and is contained in a coordinate chart in N . Then the normal bundle $T^\perp N$ is trivial over D and we can choose an orthonormal basis $e_1(x), \dots, e_d(x)$ in $T_x^\perp N$ smoothly depending on $x \in D$. Then the exponential map

$$q = f(x, u), \quad u = u_1 e_1 + \dots + u_d e_d, \quad (3.4)$$

defines coordinates $x \in D$, $u \in B_\rho = \{u \in \mathbb{R}^d : |u| < \rho\}$, in U_ρ . Then $q \in \Sigma_\rho$ when $u \in S_\rho = \partial B_\rho$. We denote by $v \in \mathbb{R}^d$ the momentum conjugate to u and by y the momentum conjugate to $x \in D$. The a trajectory of the Hamiltonian system is represented by $z(t) = (x(t), y(t)), u(t), v(t)$.

¹⁰We denote by c, C several large fixed constants.

Let F_0 be the Hamiltonian (3.5) corresponding to the Hamiltonian H_0 :

$$F_0(x, y) = \frac{1}{2} \langle A_0(x)(y - a_0(x)), y - a_0(x) \rangle + W_0(x), \quad (3.5)$$

and let

$$\mathcal{K} = \{(x, y) \in T^*D : F_0(x, y) \leq E - \varepsilon\} \subset \mathcal{M}_E. \quad (3.6)$$

If $\rho > 0$ is small enough, for any $z_0 = (x_0, y_0) \in \mathcal{K}$ and any $u_- \in S_\rho = \partial B_\rho$ there is $t_- > 0$ and a trajectory $\gamma_- = \gamma_-(z_0, u_-) : [0, t_-] \rightarrow U_\rho$ with $H = E$ satisfying the boundary conditions

$$z(0) = z_0, \quad u(0) = 0, \quad u(t_-) = u_-. \quad (3.7)$$

Similarly, for any $u_+ \in S_\rho$ there is $t_+ < 0$ and a trajectory $\gamma_+ = \gamma_+(z_0, u_+) : [t_+, 0] \rightarrow U_\rho$ with $H = E$ satisfying the boundary conditions

$$z(0) = z_0, \quad u(0) = 0, \quad u(t_+) = u_+. \quad (3.8)$$

The concatenation $\gamma_+ \cdot \gamma_-$ is a reflection trajectory of the degenerate billiard with collision point x_0 and tangent collision momentum y_0 .

Indeed, for $u = 0$ we have

$$H_0(x, y, 0, v) = F_0(x, y) + \frac{1}{2} |\dot{u}|^2 = E.$$

For a solution of the Hamiltonian system with the initial condition $z(0) = z_0$, $u(0) = 0$ and energy E , we have

$$u(t) = u(z_0, \dot{u}(0), t) = t\dot{u}(0) + O(t^2), \quad |\dot{u}(0)| = \sqrt{2(E - F_0(z_0))} = \nu(z_0).$$

For small $\rho > 0$ the equation $u(t_\pm) = u_\pm$ can be solved for

$$\begin{aligned} t_\pm &= t_\pm(z_0, u_\pm) = \mp \frac{\rho}{\nu(z_0)} + O(\rho^2), \\ \dot{u}(0) &= \dot{u}(z_0, u_\pm) = \mp \nu(z_0) e_\pm + O(\rho), \quad u_\pm = \rho e_\pm, \quad |e_\pm| = 1. \end{aligned}$$

Here $O(\rho)$ means a function of the form $\rho h(z_0, e_\pm, \rho)$ where h is C^1 bounded as $\rho \rightarrow 0$. The corresponding trajectories γ_\pm satisfy (3.8)–(3.7).

In local coordinates in D , we have $\gamma_\pm(t_\pm) = f(x_\pm, u_\pm)$, where

$$x_\pm = \xi_\pm(z_0, u_\pm) = x_0 + t_\pm \dot{x}(0) + O(\rho^2), \quad \dot{x}(0) = A_0(x_0)(y_0 - a(x_0)). \quad (3.9)$$

To prove Proposition 3.1, for given $q_\pm = f(x_\pm, u_\pm)$ such that $(q_+, q_-) \in P_\rho$, we need to find $z_0 = (x_0, y_0)$ such that

$$\xi_\pm(z_0, u_\pm) = x_\pm. \quad (3.10)$$

We have $|x_+ - x_-| \leq c\rho$ with $c > 0$ independent of ρ . Using (3.9), equations (3.10) can be rewritten as

$$x_0 = \frac{1}{2}(x_+ + x_-) + O(\rho^2), \quad y_0 = a(x_0) - \frac{\nu(z_0)}{2\rho} A_0^{-1}(x_0)(x_+ - x_-) + O(\rho).$$

For small $\rho > 0$, equations (3.10) satisfy the condition of the implicit function theorem and so they can be solved for $(x_0, y_0) = z_0(q_+, q_-)$.

Proposition 3.1 is proved. \square

Next we formulate a similar local connection result for the system $(M \setminus N, H_\mu = E)$ with Newtonian singularities. The connection trajectory will be close to the reflection trajectory $\gamma_+ \cdot \gamma_-$ of the degenerate billiard $(M, N, H_0 = E)$ in Proposition 3.1. We need another restriction on the points q_\pm we try to connect. We write it in local coordinates defined in (3.4).

Fix small $\delta > 0$ and let

$$Q_\rho = \{(q_+, q_-) \in P_\rho : q_\pm = f(x_\pm, u_\pm), |u_+ + u_-| \geq \delta\rho\}. \quad (3.11)$$

Thus we do not want the points q_\pm to be nearly opposite with respect to N .

Theorem 3.1. *Let $\rho = \rho(\delta, C, D) > 0$ be sufficiently small. There exists $\mu_0 > 0$ such that for all $(q_+, q_-) \in Q_\rho$ and $\mu \in I_{\mu_0} = (-\mu_0, 0) \cup (0, \mu_0)$:*

- *There exists a unique (up to a time shift) trajectory α_μ of system $(M \setminus N, H_\mu = E)$ joining q_+ and q_- in $B_r(x_0)$, where $x_0 = \xi(q_+, q_-)$.*
- *α_μ smoothly depends on $(q_+, q_-, \mu) \in Q_\rho \times I_{\mu_0}$ and uniformly converges (as a nonparametrized curve) as $\mu \rightarrow 0$ to the billiard trajectory $\gamma_+ \cdot \gamma_-$ in Proposition 3.1.*
- *The minimal distance $d(\alpha_\mu, N)$ is attained at a point $q_\mu = f(x_\mu, u_\mu)$, which converges to $x_0 = \xi(q_+, q_-)$ as $\mu \rightarrow 0$:*

$$|u_\mu| \leq c|\mu|, \quad d(x_\mu, x_0) \leq c|\mu \ln |\mu||.$$

- *The Maupertuis action of α_μ has the form*

$$J_\mu(\alpha_\mu) = \int_{\alpha_\mu} p dq = R_\mu(q_+, q_-) = R_0(q_+, q_-) + O(\mu \ln |\mu|), \quad (3.12)$$

where $R_0(q_+, q_-)$ is the action (3.3) of the billiard trajectory $\gamma_+ \cdot \gamma_-$ and $O(\mu \ln |\mu|)$ means a function h such that

$$\|h\|_{C^2(Q_\rho)} \leq c|\mu \ln |\mu||$$

with a constant c independent of μ .

Theorem 3.1 implies that the symplectic map $\mathcal{P}_\mu : (q_+, p_+) \rightarrow (q_-, p_-)$ of $T^*\Sigma_\rho$, which has no limit as $\mu \rightarrow 0$, does have a smooth limit if represented as a Lagrangian relation with the generating function R_μ .

Remark 3.1. *For the attracting force ($\mu > 0$) the connecting trajectory α_μ in Theorem 3.1 may have a regularizable collision with N (although the set of*

(q_+, q_-) with this property is negligible). To avoid this, we have to replace Q_ρ with the set

$$\hat{Q}_\rho = \{(q_-, q_+) \in Q_\rho : q_\pm = f(x_\pm, u_\pm), |u_+ - u_-| \geq \delta\rho\}.$$

If $(q_+, q_-) \in \hat{Q}_\rho$, then the billiard trajectory $\gamma_+ \cdot \gamma_-$ satisfies the no straight reflection condition (2.14) at x_0 . Then the shadowing orbit α_μ will satisfy

$$c_1\mu \leq d(\alpha_\mu, N) \leq c_2\mu, \quad c_{1,2} > 0.$$

Theorem 3.1 is a generalization of the result proved in [11]. We will deduce it from the following Theorem 3.2. Let $\mathcal{K} \subset \mathcal{M}_E$ be the set (3.6).

Theorem 3.2. Fix $\delta > 0$. Let $\rho > 0$ be sufficiently small. There exists $\mu_0 > 0$ such that for all $\mu \in I_{\mu_0}$, any $z_0 = (x_0, y_0) \in \mathcal{K}$ and any $u_\pm \in S_\rho$ such that $|u_+ + u_-| \geq \delta\rho$:

- There exists a trajectory $\gamma_\mu : [t_+, t_-] \rightarrow N_\rho$, $t_+ < 0 < t_-$, of system $(M \setminus N, H_\mu = E)$ satisfying the initial-boundary conditions $u(t_\pm) = u_\pm$, $z(0) = z_0$.
- γ_μ smoothly depends on $(z_0, u_+, u_-, \mu) \in \mathcal{K} \times S_\rho^2 \times I_{\mu_0}$ and converges, as $\mu \rightarrow 0$, to a trajectory $\gamma_+ \cdot \gamma_-$ of the degenerate billiard having a reflection from N at x_0 with the tangent momentum y_0 .
- The Maupertuis action of γ has the form

$$J_\mu(\gamma_\mu) = \psi_+(z_0, u_+) + \psi_-(z_0, u_-) + O(\mu \ln |\mu|), \quad (3.13)$$

where $\psi_\pm(z_0, u_\pm)$ are the Maupertuis actions of the trajectories $\gamma_\pm(z_0, u_\pm)$ of the Hamiltonian system $(M, H_0 = E)$ satisfying the boundary conditions (3.8)–(3.7).

- The end points of γ_μ satisfy

$$x(t_\pm) = x_\mu^\pm(z_0, u_+, u_-) = \xi_\pm(z_0, u_\pm) + O(\mu \ln |\mu|). \quad (3.14)$$

Here $O(\mu \ln |\mu|)$ means a function which is uniformly C^1 bounded on $\mathcal{K} \times S_\rho^2$ for $\mu \in I_{\mu_0}$ by $c|\mu \ln |\mu||$.

- If $\mu < 0$, or $\mu > 0$ and $|u_+ - u_-| \geq \delta\rho$, then

$$\mu c_1 \leq d(\gamma_\mu, N) = \min |u(t)| \leq \mu c_2, \quad 0 < c_1 < c_2. \quad (3.15)$$

Let us deduce Theorem 3.1 from Theorem 3.2. We have $q_\pm = f(x_\pm, u_\pm)$, $u_\pm \in S_\rho$, where $d(x_+, x_-) \leq c\rho$ and $|u_+ + u_-| \geq \delta\rho$. We need to find $z_0 \in \mathcal{K}$ such that the trajectory γ_μ in Theorem 3.2 corresponding to u_\pm and $z_0 \in \mathcal{K}$ satisfies $x_\mu^\pm(z_0, u_-, u_+) = x_\pm$.

For $\mu = 0$ this is done in the proof of Proposition 3.1 and $z_0 = z_0(q_+, q_-)$ was obtained as a nondegenerate solution of equations (3.10). Since the implicit function theorem worked for $\mu = 0$, by (3.14), for small μ it will work also here. \square

Theorem 3.2 is proved (for $d \leq 3$) in section 4.

3.2 Proof of Theorem 2.1

The idea of the proof is to represent the shadowing trajectory of system $(M \setminus N, H_\mu = E)$ as a critical point of a functional Φ_μ which is nonsingular as $\mu \rightarrow 0$.

Let $\gamma^0 = (\gamma_j^0)$ be a nondegenerate n -periodic collision chain of the degenerate billiard $(M, N, H_0 = E)$. Suppose the collision orbit γ_j^0 connects the points $x_j^0 \in N$ and $x_{j+1}^0 \in N$. There is a n -periodic sequence $\mathbf{k} = (k_j)$ such that $\mathbf{x}^0 = (x_j^0)$ is a nondegenerate critical point of the function (2.12).

Since $x_j^0 \in \mathcal{D}_E$, there exists $r > 0$ such that $B_r(x_j^0) \Subset \mathcal{D}_E$ for all j . Set $D_j = B_r(x_j^0) \cap N$ and $D = \cup D_j$.

Take small $\rho > 0$. Suppose that the collision orbit γ_j^0 crosses Σ_ρ at the points s_j^- near x_j^0 and s_{j+1}^+ near x_{j+1}^0 . Since γ_j^0 is not tangent to N at the end points, taking ρ small enough we may assume that there is a constant $C > 0$, independent of ρ , such that

$$d(s_j^+, s_j^-) < C\rho.$$

Hence $(s_j^+, s_j^-) \in P_\rho$, where P_ρ is the set (3.2) corresponding to C, D . We take $\rho > 0$ so small that Proposition 3.1 holds in P_ρ . There is $\varepsilon > 0$ such that for

$$q_j^\pm \in B_j^\pm = \Sigma_\rho \cap B_\varepsilon(s_j^\pm),$$

we have $(q_j^+, q_j^-) \in P_\rho$. Then the action function R_0 in Proposition 3.1 is defined on $B_j^+ \times B_j^-$.

In the coordinates $x \in D_j$, $u \in B_\rho$ in a neighborhood of x_j^0 , we have $s_j^\pm = f(x_j^\pm, u_j^\pm)$. The jump condition implies that there is $\delta > 0$ such that $|u_j^- + u_j^+| \geq 2\delta\rho$. Then if $\varepsilon > 0$ is small enough, $q_j^\pm \in B_j^\pm$ implies $(q_j^+, q_j^-) \in Q_\rho$, where Q_ρ is the set (3.11) corresponding to C, D, δ . By Theorem 3.1, if $\rho > 0$ is small enough, there is $\mu_0 > 0$ such that for $\mu \in I_{\mu_0} = (-\mu_0, 0) \cup (0, \mu_0)$ there exists an orbit $\alpha_j = \alpha_j(q_j^+, q_j^-, \mu)$ of system $(M \setminus N, H_\mu = E)$ joining q_j^+ and q_j^- in $N_\rho \cap B_r(x_j^0)$. Its action $J_\mu(\alpha_j) = R_\mu(q_j^+, q_j^-)$ is given by (3.12).

Since the points x_j^0 and x_{j+1}^0 are non-conjugate along γ_j^0 , for small $\rho > 0$ the points s_j^- and s_{j+1}^+ are not conjugate along the corresponding segment of γ_j^0 . Hence there exist $\varepsilon > 0$ and $\mu_0 > 0$ such that for any $\mu \in (-\mu_0, \mu_0)$, any points $q_j^- \in B_j^-$ and $q_{j+1}^+ \in B_{j+1}^+$ are joined by a unique trajectory $\beta_j = \beta_j(q_j^-, q_{j+1}^+, \mu)$ of system $(M \setminus N, H_\mu = E)$ which is close to γ_j^0 . Let $F_j(q_j^-, q_{j+1}^+, \mu) = J_\mu(\beta_j)$ be its Maupertuis action.

Consider the function

$$\Phi_\mu(\mathbf{q}) = \sum_{j=1}^n (F_j(q_j^-, q_{j+1}^+, \mu) + R_\mu(q_j^+, q_j^-)), \quad q_{n+1}^\pm = q_1^\pm, \quad (3.16)$$

where

$$\mathbf{q} = (q_1^+, q_1^-, \dots, q_n^+, q_n^-) \in \mathcal{B} = B_1^+ \times B_1^- \times \dots \times B_n^+ \times B_n^-. \quad (3.17)$$

Then $\Phi_\mu(\mathbf{q})$ is the Maupertuis action $J_\mu(\hat{\gamma})$ of the concatenation $\hat{\gamma}$ of the trajectories α_j, β_j defined above. This is a broken trajectory with momentum discontinuous at q_j^\pm .

Lemma 3.1. *If $\mathbf{q} \in \mathcal{B}$ is a critical point of Φ_μ , then the concatenation $\hat{\gamma}$ is a smooth periodic trajectory of system $(M \setminus N, H_\mu = E)$.*

Indeed, by Hamilton's first variation formula,

$$\delta\Phi_\mu(\mathbf{q}) = \delta J_\mu(\hat{\gamma}) = \sum_{j=1}^n (\langle \Delta p_j^+, \delta q_j^+ \rangle + \langle \Delta p_j^-, \delta q_j^- \rangle) = 0, \quad \delta q_j^\pm \in T_{q_j^\pm} \Sigma_\rho,$$

where Δp_j^\pm is the jump of the momentum at q_j^\pm . Hence $\Delta p_j^\pm \perp T_{q_j^\pm} \Sigma_\rho$. Since the Hamiltonian $H_\mu = E$ has no jump, and Σ_ρ is a hypersurface, this implies $\Delta p_j^\pm = 0$.

Indeed, let u_j be the initial velocity of α_j at q_j^+ and v_j the final velocity of β_{j-1} at q_j^+ . Then $\Delta v_j = u_j - v_j$ is orthogonal to $T_{q_j^+} \Sigma_\rho$ with respect to the Riemannian metric and

$$\|u_j\|^2 = \|v_j\|^2 = 2(E - W_\mu(q_j^+)).$$

This implies that either $u_j = v_j$ and $\Delta v_j = 0$, so the concatenation $\beta_{j-1} \cdot \alpha_j$ is smooth at q_j^+ , or the concatenation has an elastic reflection from Σ_ρ , and then $\Delta v_j \neq 0$. The second case is impossible since $\alpha_j \subset N_\rho$, so its velocity u_j at q_j^+ points outside Σ_ρ , and the velocity v_j of β_j at q_j^+ is close to the velocity of γ_{j-1}^0 at s_j^+ so it also points outside Σ_ρ .

Hence the concatenation $\hat{\gamma}$ is smooth at q_j^+ . Similarly for q_j^- . \square

Let us show that for $\mu = 0$ the function Φ_0 has a nondegenerate critical point $\mathbf{q}^0 = (s_1^+, s_1^-, \dots, s_n^+, s_n^-)$. Indeed, consider the function

$$\Psi(\mathbf{q}, \mathbf{x}) = \sum_{j=1}^n (F_j(q_j^-, q_{j+1}^+, 0) + S(q_{j+1}^+, x_{j+1}) + S(x_j, q_j^-)), \quad (3.18)$$

where $q_j^\pm \in B_j^\pm$, $x_j \in D_j$ and $q_{n+1}^\pm = q_1^\pm$, $x_{n+1} = x_1$.

By Proposition 3.1, for fixed $\mathbf{q} = (q_1^+, q_1^-, \dots, q_n^+, q_n^-) \in \mathcal{B}$, the function $\mathbf{x} \rightarrow \Psi(\mathbf{q}, \mathbf{x})$ has a nondegenerate critical point $\mathbf{x} = \mathbf{x}(\mathbf{q}) \in N^n$, $x_j = \xi(q_j^-, q_j^+)$ and by (3.3), the critical value is

$$\begin{aligned} \Psi(\mathbf{q}, \mathbf{x}(\mathbf{q})) &= \text{Crit}_{\mathbf{x}} \sum_{j=1}^n (F_j(q_j^-, q_{j+1}^+, 0) + R_0(q_j^+, x_j, q_j^-)) \\ &= \sum_{j=1}^n (F_j(q_j^-, q_{j+1}^+, 0) + R_0(q_j^+, q_j^-)) = \Phi_0(\mathbf{q}). \end{aligned}$$

For $\mathbf{q} = \mathbf{q}^0$, we have $\mathbf{x}(\mathbf{q}^0) = \mathbf{x}^0$. On the other hand, for fixed \mathbf{x} , the function $\mathbf{q} \rightarrow \Psi(\mathbf{q}, \mathbf{x})$ has a nondegenerate critical point $\mathbf{q} = \mathbf{q}(\mathbf{x})$ of the form (3.17). The critical value is

$$\Psi(\mathbf{q}(\mathbf{x}), \mathbf{x}) = \mathcal{A}_{\mathbf{k}}^{(n)}(\mathbf{x}) = \sum_{j=1}^n J(\gamma_j),$$

where γ_j is a trajectory of system $(M, H_0 = E)$ joining $x_j, x_{j+1} \in N$ and crossing Σ_ρ at the points q_j^-, q_{j+1}^+ . By the assumption, the function $\mathcal{A}_{\mathbf{k}}^{(n)}(\mathbf{x})$ has a nondegenerate critical point \mathbf{x}^0 . Then $(\mathbf{q}^0, \mathbf{x}^0)$, $\mathbf{q}^0 = \mathbf{q}(\mathbf{x}^0)$, is a nondegenerate critical point of Ψ . Hence \mathbf{q}^0 is a nondegenerate critical point of Φ_0 .

By (3.13),

$$\Phi_\mu(\mathbf{q}) = \Phi_0(\mathbf{q}) + O(\mu \ln |\mu|).$$

By the implicit function theorem, for small $\mu \neq 0$, $\Phi_\mu(\mathbf{q})$ has a nondegenerate critical point near \mathbf{q}^0 which defines a periodic orbit of the system $(M \setminus N, H_\mu)$ shadowing the chain γ^0 . \square

If there is a symmetry group $\Phi_\theta : M \rightarrow M$, then everything will be invariant under Φ_θ , and we obtain a proof of Theorem 2.4.

3.3 Proof of Theorem 2.3

It is similar to the proof of Theorem 2.1. We only need to check uniformity. Let $\Lambda \subset K^{\mathbb{Z}} \times N^{\mathbb{Z}}$ be a compact hyperbolic \mathcal{T} -invariant set of admissible trajectories of the DLS.

There exist a finite collection $\{\Omega_k\}_{k \in I}$ of compact¹¹ sets of collision orbits $\gamma : [t_-, t_+] \rightarrow M$ such that collision chains $(\gamma_j)_{j \in \mathbb{Z}}$ corresponding to trajectories $(\mathbf{k}, \mathbf{x}) \in \Lambda$ are concatenations of collision orbits $\gamma_j \in \Omega_{k_j}$.

Collision orbits $\gamma \in \Omega_k$ join pairs of nonconjugate points $x_-(\gamma) \in N$ and $x_+(\gamma) \in N$ which form compact sets

$$X_k^\pm = \{x_\pm(\gamma) : \gamma \in \Omega_k\} \subset N.$$

Take open sets $D_k^\pm \Subset \mathcal{D}_E \cap N$ such that $X_k^\pm \Subset D_k^\pm$ for all $k \in I$. Set $D = \bigcup (D_k^+ \cup D_{k'}^-)$. Take sufficiently small $\rho > 0$ and let Σ_ρ be the corresponding set (3.1).

For any $\gamma \in \Omega_k$ let $s_-(\gamma)$ and $s_+(\gamma)$ be the first and last intersection points with Σ_ρ . By the definition of a collision orbit (2.4), the angles between initial and final velocities $v_\pm(\gamma)$ and N , and the collision speeds $\|v_\pm(\gamma)\|$ are bounded away from 0. Hence there exists $c > 0$, independent of ρ and $\gamma \in \Omega_k$, such that $d(x_\pm(\gamma), s_\pm(\gamma)) \leq c\rho$. Then

$$Y_k = \{(s_-(\gamma), s_+(\gamma)) \in \Sigma_\rho^2 : \gamma \in \Omega_k\}$$

¹¹Topology on the set of collision orbits γ is defined by reparametrizing γ proportionally to the arc length (in the metric $\|\cdot\|$), and using the topology in $C^0([0, 1], M)$.

is a compact set contained in \mathcal{D}_E . We can assume that for any $(q_-, q_+) \in Y_k$ there is unique $\gamma(q_-, q_+) \in \Omega_k$ such that $s_\pm(\gamma) = q_\pm$.

There is $\mu_0 > 0$ such that for any $\mu \in (-\mu_0, \mu_0)$, any collision orbit $\gamma \in \Omega_k$, any pair of points q_-, q_+ in the set

$$\hat{Y}_k = \{(q_-, q_+) \in \Sigma_\rho^2 : d((q_-, q_+), Y_k) \leq \varepsilon\}.$$

are joined by a trajectory $\beta_\mu = \beta_\mu(q_-, q_+, k)$ of system $(M \setminus N, H_\mu = E)$ which is close to $\gamma(q_-, q_+)$. This follows from compactness of Ω_k and nonconjugacy of $s_\pm(\gamma)$ along $\gamma \in \Omega_k$. Then the Maupertuis action

$$F_k(q_-, q_+, \mu) = J_\mu(\beta_\mu)$$

is a smooth function on \hat{Y}_k .

Every collision chain corresponding to a trajectory in Λ is a concatenation of collision orbits in $\{\Omega_k\}$. Let $\Pi_{k,k'} \subset \Omega_k \times \Omega_{k'}$, $(k, k') \in I^2$, be the compact set of all pairs $\gamma \in \Omega_k$, $\gamma' \in \Omega_{k'}$ of neighbor collision orbits in such concatenations. Let $s_+(\gamma)$, $s_-(\gamma')$ be the corresponding points in Σ_ρ . Set

$$V_{kk'} = \{(s_+(\gamma), s_-(\gamma')) : (\gamma, \gamma') \in \Pi_{k,k'}\}.$$

There is a constant $c > 0$ such that $d(q_+, q_-) \leq 2c\rho$ for all $(q_+, q_-) \in V_{k,k'}$. If we take $C > 2c$, then $V_{kk'} \subset P_\rho$, where P_ρ is the set (3.2) corresponding to D and the constant C .

Let

$$s_\pm(\gamma) = f(x_\pm(\gamma), u_\pm(\gamma)), \quad x_\pm(\gamma) \in D_k^\pm, \quad |u_\pm(\gamma)| = \rho.$$

By the jump condition and compactness of Λ , if $\delta > 0$ is small enough,

$$|u_+(\gamma) + u_-(\gamma')| \geq \delta\rho \quad \text{for all } (\gamma, \gamma') \in \Pi_{k,k'}.$$

Let $Q_\rho \subset P_\rho$ be the set (3.11) corresponding to D and the constants $C, \delta > 0$. Then $V_{kk'} \subset Q_\rho$. There exist $\varepsilon > 0$ such that $d((q_+, q_-), V_{kk'}) < \delta$ implies $(q_+, q_-) \in Q_\rho$.

Remark 3.2. *If also the no straight reflection condition holds, then $|u_+(\gamma) - u_-(\gamma')| \geq \delta\rho$, and so $(s_+(\gamma), s_-(\gamma')) \in \hat{Q}_\rho$. Then $V_{kk'} \subset \hat{Q}_\rho$.*

We take $\rho > 0$ so small that Theorem 3.1 holds in Q_ρ . Then there is $\mu_0 > 0$ such that for any $\mu \in I_{\mu_0}$, the points $(q_+, q_-) \in Q_\rho$ can be joined by a trajectory $\alpha_\mu = \alpha_\mu(q_+, q_-)$ with action $J_\mu(\alpha_\mu) = R_\mu(q_+, q_-)$.

Let $(\mathbf{k}, \mathbf{x}^0) \in \Lambda$ be a trajectory of the DLS, and let $\gamma^0 = (\gamma_j^0)_{j \in \mathbb{Z}}$, $(\gamma_j^0, \gamma_{j+1}^0) \in \Pi_{k_j k_{j+1}}$, be the corresponding collision chain, where γ_j^0 connects the points $x_j^0, x_{j+1}^0 \in D$ and intersects Σ_ρ at the points $s_j^- = s_-(\gamma_j)$ and $s_{j+1}^+ = s_+(\gamma_j)$. Then $(s_j^-, s_{j+1}^+) \in \hat{Y}_{k_j}$ and $(s_j^+, s_{j+1}^-) \in \hat{Y}_{k_{j+1}}$.

As in (3.16), consider the formal functional

$$\Phi_\mu(\mathbf{q}) = \sum_{j \in \mathbb{Z}} (F_{k_j}(q_j^-, q_{j+1}^+, \mu) + R_\mu(q_j^+, q_j^-)), \quad \mathbf{q} = (q_j^-, q_j^+)_{j \in \mathbb{Z}},$$

where

$$(q_j^-, q_{j+1}^+) \in \hat{Y}_{k_j}, \quad (q_j^+, q_j^-) \in Q_\rho.$$

The functional depends on $\mathbf{k} \in I^\mathbb{Z}$, but we do not show it in the notation. As in the proof of Theorem 2.1, $\Phi_\mu(\mathbf{q})$ is the action of an infinite concatenation $\hat{\gamma}$ of trajectories $\alpha_\mu(q_j^+, q_j^-)$ and $\beta_\mu(q_j^-, q_{j+1}^+, k_j)$. The derivative $D\Phi_\mu(\mathbf{q}) = \Gamma_\mu(\mathbf{q})$ makes sense, so critical points are well defined. As in the proof of Theorem 2.1, critical points of Φ_μ correspond to trajectories of system $(M \setminus N, H_\mu = E)$ shadowing the collision chain γ^0 .

Let us show that for $\mu = 0$ the functional Φ_0 has a uniformly nondegenerate critical point $\mathbf{q}^0 = \mathbf{q}^0(\mathbf{x}^0)$:

$$\Gamma_0(\mathbf{q}^0) = 0, \quad \|D\Gamma_0(\mathbf{q}^0)^{-1}\|_\infty \leq C_2, \quad \Gamma_0(\mathbf{q}) = D\Phi_0(\mathbf{q}), \quad (3.19)$$

with $C_2 = C_2(\Lambda)$ independent of the trajectory $(\mathbf{k}, \mathbf{x}^0) \in \Lambda$. The l_∞ norm can be defined by using the Riemannian metric on M to identify $D\Gamma_0(\mathbf{q})$ with a linear operator on an l_∞ Banach space

$$E_1 \subset \prod_{j \in \mathbb{Z}} (T_{q_j^+} \Sigma_\rho \times T_{q_j^-} \Sigma_\rho)$$

with the l_∞ norm. A simpler option is to use local coordinates.

We can introduce coordinate charts O_j^\pm on Σ_ρ containing the points s_j^\pm by using e.g. the exponential maps $\exp_{s_j^\pm} : T_{s_j^\pm} \Sigma_\rho \rightarrow \Sigma_\rho$. Then we identify O_j^\pm with a ball $\{q \in \mathbb{R}^{m-1} : |q - s_j^\pm| < \varepsilon\}$, $m = \dim M$. Then $\mathbf{q} = (q_j^-, q_j^+)_{j \in \mathbb{Z}}$ is represented by a point in a ball

$$Z_1 = \{(\mathbf{q}^-, \mathbf{q}^+) : \|\mathbf{q}^\pm - \mathbf{s}^\pm\|_\infty < \varepsilon\}$$

in the Banach space $E_1 = l_\infty(\mathbb{R}^{m-1})$. Thus Γ_μ is now a map $\Gamma_\mu : Z_1 \subset E_1 \rightarrow E_1$, and $D\Gamma_\mu(\mathbf{q})$ is a bounded operator in E_1 .

Similarly we introduce local coordinates in a ball $D_j = B_\varepsilon(x_j^0) \cap N \subset D_{k_j}^+ \cap D_{k_{j+1}}^-$ by using e.g. the exponential map $\exp_{x_j^0} : T_{x_j^0} N \rightarrow N$. Then we identify $x_j \in D_j$ with a point in the ball¹² $\{x \in \mathbb{R}^{n_j} : |x - x_j^0| < \varepsilon\}$. Then for a trajectory (\mathbf{k}, \mathbf{x}) we can regard $\mathbf{x} = (x_j)_{j \in \mathbb{Z}}$ as a point in a ball $Z_2 = \{\mathbf{x} : \|\mathbf{x} - \mathbf{x}^0\|_\infty < \varepsilon\}$ in the l_∞ Banach space

$$E_2 = \{\mathbf{x} \in \prod_{j \in \mathbb{Z}} \mathbb{R}^{n(j)} : \|\mathbf{x}\|_\infty = \sup |x_j| < \infty\}.$$

To show that $D\Gamma_0(\mathbf{q}^0) : E_1 \rightarrow E_1$ is invertible, as in (3.18), consider the functional

$$\Psi(\mathbf{q}, \mathbf{x}) = \sum_{j \in \mathbb{Z}} (F_{k_j}(q_j^-, q_{j+1}^+, 0) + S(q_{j+1}^+, x_{j+1}) + S(x_j, q_j^-)),$$

¹²Recall that components of N may have different dimensions.

where $(\mathbf{q}, \mathbf{x}) \in Z = Z_1 \times Z_2$. The functional is formal, but its derivatives

$$D_{\mathbf{q}}\Psi(\mathbf{q}, \mathbf{x}) = G_1(\mathbf{q}, \mathbf{x}), \quad D_{\mathbf{x}}\Psi(\mathbf{q}, \mathbf{x}) = G_2(\mathbf{q}, \mathbf{x}),$$

are well defined. Then $G = (G_1, G_2)$ is a C^1 map from an open set $Z = Z_1 \times Z_2 \subset E = E_1 \times E_2$ to E and

$$\|G(\mathbf{q}, \mathbf{x})\|_{\infty} \leq C_3, \quad \|DG(\mathbf{q}, \mathbf{x})\|_{\infty} \leq C_3, \quad (\mathbf{q}, \mathbf{x}) \in Z,$$

where the constant $C_3 = C_3(\Lambda)$ is independent of the trajectory $(\mathbf{k}, \mathbf{x}^0)$.

By Proposition 3.1, the equation $G_2(\mathbf{q}, \mathbf{x}) = 0$ has a nondegenerate solution $\mathbf{x}(\mathbf{q})$ such that

$$G_2(\mathbf{q}, \mathbf{x}(\mathbf{q})) = 0, \quad \|D_{\mathbf{x}}G_2(\mathbf{q}, \mathbf{x}(\mathbf{q}))^{-1}\|_{\infty} \leq C_4 = C_4(\Lambda).$$

Indeed, $\mathbf{x}(\mathbf{q}) = (x_j(\mathbf{q}))$ where $x_j(\mathbf{q}) = \xi(q_{j-1}^+, q_j^-)$ is a nondegenerate critical point of the function $x \rightarrow R(q_j^+, x, q_j^-)$. The operator $D_{\mathbf{x}}G_2$ is block diagonal, so nondegeneracy implies that the inverse is l_{∞} bounded. We have $G_1(\mathbf{q}, \mathbf{x}(\mathbf{q})) = \Gamma_0(\mathbf{q})$.

Now we use the following lemma [4] which is a version of the Lyapunov–Schmidt reduction.

Lemma 3.2. *Let $E = E_1 \times E_2$ be Banach spaces and let $G = (G_1, G_2) : Z \rightarrow E$ be a C^1 map of an open set $Z = Z_1 \times Z_2 \subset E$. Suppose that there is $C > 0$ such that*

$$\|DG(\mathbf{q}, \mathbf{x})\| \leq C, \quad \|(D_{\mathbf{x}}G_2(\mathbf{q}, \mathbf{x}))^{-1}\| \leq C \quad \text{for all } (\mathbf{q}, \mathbf{x}) \in Z.$$

Let $G(\mathbf{q}^0, \mathbf{x}^0) = 0$ and let $\mathbf{x} = \mathbf{x}(\mathbf{q})$, $\mathbf{q} \in Z_1$, be a solution of $G_2(\mathbf{q}, \mathbf{x}) = 0$ such that $\mathbf{x}(\mathbf{q}^0) = \mathbf{x}^0$. Set $\Gamma(\mathbf{q}) = G_1(\mathbf{q}, \mathbf{x}(\mathbf{q}))$. Then $D\Gamma(\mathbf{q})$ and $DG(\mathbf{q}, \mathbf{x}(\mathbf{q}))$ are invertible simultaneously and there exists a constant $c = c(C) > 0$ such that

$$\|(D\Gamma(\mathbf{q}))^{-1}\| \leq \|(DG(\mathbf{q}, \mathbf{x}(\mathbf{q})))^{-1}\| \leq c(1 + \|(D\Gamma(\mathbf{q}))^{-1}\|).$$

Thus (3.19) holds. We conclude that \mathbf{q}^0 is a uniformly l_{∞} -nondegenerate critical point of Φ_0 independently of a trajectory $(\mathbf{k}, \mathbf{x}^0) \in \Lambda$. By (3.13),

$$\|D\Gamma_{\mu}(\mathbf{q}) - D\Gamma_0(\mathbf{q})\|_{\infty} \leq C_5 |\mu \ln |\mu||, \quad C_5 = C_5(\Lambda).$$

The proof of Theorem 2.3 is completed by using a uniform version of the implicit function theorem.

4 Regularization

In this section we prove Theorem 3.2 for $d = \text{codim } N \leq 3$. For $d \leq 2$ we use the Levi-Civita regularization, and for $d = 3$ the KS-regularization. For $d \geq 4$ a different method is needed.

Let $f_\mu : T^\perp N \rightarrow M$ the exponential map corresponding to the Riemannian metric $\|\cdot\|_\mu$. As in (3.4), we assume that $D \Subset N$ is contractible and choose an orthonormal basis $e_1(x), \dots, e_d(x)$ in $T_x^\perp N$ smoothly depending on $x \in D$. The map

$$D \times B_\rho \rightarrow U_\rho, \quad q = f_\mu(x, u), \quad u = u_1 e_1(x) + \dots + u_d e_d(x), \quad (4.1)$$

defines semigeodesic coordinates $x \in D$, $u \in B_\rho = \{v \in \mathbb{R}^d : |v| < \rho\}$, in $U = U_\rho$. The Riemannian metric $\|\cdot\|_\mu$ in U has the form

$$\|\dot{q}\|_\mu^2 = \langle \mathcal{A}(x, u, \mu) \dot{x}, \dot{x} \rangle + |\dot{u}|^2 + \langle \mathcal{B}(x, u, \mu) \dot{u}, \dot{u} \rangle + \langle \mathcal{C}(x, u, \mu) \dot{x}, \dot{u} \rangle, \quad (4.2)$$

where \mathcal{A} is positive definite and¹³

$$\mathcal{B}(x, u, \mu) = O_2(u), \quad \mathcal{C}(x, u, \mu) = O_1(u).$$

By the properties of the exponential map, $d_\mu(q, N) = |u|$.

Let $y \in T_x^* D$, $v \in \mathbb{R}^d$, be the momenta conjugate to x, u , so that

$$\langle p, dq \rangle = \langle y, dx \rangle + \langle v, du \rangle.$$

Then

$$\|p\|_\mu^2 = \langle A(x, u, \mu) y, y \rangle + |v|^2 + \langle B(x, u, \mu) v, v \rangle + \langle C(x, u, \mu) y, v \rangle, \quad (4.3)$$

where

$$B(x, u, \mu) = O_2(u), \quad C(x, u, \mu) = O_1(u), \quad A(x, 0, \mu) = \mathcal{A}^{-1}(x, 0, \mu).$$

The gyroscopic 1-form is

$$\langle w_\mu(q), dq \rangle = \langle a(x, u, \mu), dx \rangle + \langle b(x, u, \mu), du \rangle. \quad (4.4)$$

Without loss of generality we may assume that

$$b(x, u, \mu) = O(u). \quad (4.5)$$

Indeed,

$$\langle w_\mu(q), dq \rangle = \langle \tilde{a}(x, u, \mu), dx \rangle + \langle \tilde{b}(x, u, \mu), du \rangle + d\varphi(x, u, \mu),$$

where

$$\tilde{b} = b(x, u, \mu) - b(x, 0, \mu), \quad \varphi = \langle b(x, 0, \mu), u \rangle, \quad \tilde{a} = a(x, u, \mu) - D_x \varphi(x, u, \mu).$$

The differential $d\varphi$ can be dropped: it does not affect trajectories $q(t)$ (only the corresponding momenta $p(t)$) since it changes only the boundary terms in the action functional (2.1). The new coefficient \tilde{b} satisfies (4.5).

¹³Here $O_k(u)$ means a function whose Taylor expansion with respect to u starts with k -th order terms.

In the symplectic variables x, y, u, v the Hamiltonian (1.11) has the form

$$\begin{aligned}
H_\mu(q, p) &= \frac{1}{2} \left(\langle A(x, u, \mu)(y - a(x, u, \mu)), y - a(x, u, \mu) \rangle + |v - b(x, u, \mu)|^2 \right. \\
&\quad + \langle B(x, u, \mu)(v - b(x, u, \mu)), v - b(x, u, \mu) \rangle \\
&\quad \left. + \langle C(x, u, \mu)(y - a(x, u, \mu)), v - b(x, u, \mu) \rangle \right) \\
&\quad + W(x, u, \mu) - \frac{\mu \phi(x, u, \mu)}{|u|}.
\end{aligned} \tag{4.6}$$

Next we regularize the singularity at $u = 0$.

4.1 Codimension 2

Let $d = 2$. Then we identify $\mathbb{R}^2 = \mathbb{C}$ and use the Levi-Civita change of variables

$$u = u(\xi) = \xi^2/2, \quad |u| = |\xi|^2/2, \quad du = \xi d\xi.$$

In the real variables,

$$u(\xi) = \frac{1}{2} \Gamma(\xi) \xi, \quad \Gamma(\xi) = \begin{pmatrix} \xi_1 & -\xi_2 \\ \xi_2 & \xi_1 \end{pmatrix}.$$

The matrix Γ is orthogonal:

$$\Gamma^*(\xi) \Gamma(\xi) = \Gamma(\xi) \Gamma^*(\xi) = |\xi|^2 I_2. \tag{4.7}$$

The square map evidently satisfies $u(\xi_+) = u(\xi_-)$ iff $\xi_+ = \pm \xi_-$ and

$$u(\xi_+) = -u(\xi_-) \Leftrightarrow \langle \xi_+, \xi_- \rangle = 0, \quad |\xi_+| = |\xi_-|. \tag{4.8}$$

Let η be the momentum conjugate to ξ so that

$$\langle v, du \rangle = \langle v, \Gamma(\xi) d\xi \rangle = \langle \Gamma^*(\xi) v, d\xi \rangle = \langle \eta, d\xi \rangle.$$

Thus

$$\eta = \Gamma^*(\xi) v, \quad v = \frac{\Gamma(\xi) \eta}{|\xi|^2}, \quad |\eta| = |v| |\xi|.$$

Remark 4.1. *In the complex notation, the formulas are much simpler: e.g. $\eta = \bar{\xi} v$. But we need to write the transformation in the form which will work also for $d = 3$.*

The gyroscopic 1-form is now

$$\langle w_\mu(q), dq \rangle = \langle a(x, u(\xi), \mu), dx \rangle + \langle \hat{b}(x, \xi, \mu), d\xi \rangle,$$

where

$$\hat{b}(x, \xi, \mu) = \Gamma^*(\xi) b(x, u(\xi), \mu) = O_3(\xi).$$

We have

$$v - b(x, u, \mu) = \frac{\Gamma(\xi)\eta - |\xi|^2 b(x, u(\xi), \mu)}{|\xi|^2} = \frac{\Gamma(\xi)(\eta - \hat{b}(x, \xi, \mu))}{|\xi|^2}.$$

By (4.3),

$$\begin{aligned} \|p - w_\mu(q)\|_\mu^2 &= \langle A(x, u(\xi), \mu)(y - a(x, u(\xi), \mu)), y - a(x, u(\xi), \mu) \rangle \\ &+ \frac{|\eta - \hat{b}(x, \xi, \mu)|^2}{|\xi|^2} + \frac{\langle \hat{B}(x, \xi, \mu)(\eta - \hat{b}(x, \xi, \mu)), \eta - \hat{b}(x, \xi, \mu) \rangle}{|\xi|^4} \\ &+ \frac{\langle \hat{C}(x, \xi, \mu)(y - a(x, u(\xi), \mu)), \eta - \hat{b}(x, \xi, \mu) \rangle}{|\xi|^2} \end{aligned}$$

where

$$\hat{B}(x, \xi, \mu) = \Gamma^*(\xi)B(x, u(\xi), \mu)\Gamma(\xi) = O_6(\xi), \quad (4.9)$$

$$\hat{C}(x, \xi, \mu) = \Gamma^*(\xi)C(x, u(\xi), \mu) = O_3(\xi). \quad (4.10)$$

The equation $H_\mu = E$ takes the form

$$\begin{aligned} &\left(\frac{1}{2} \langle A(x, u(\xi), \mu)(y - a(x, u(\xi), \mu)), y - a(x, u(\xi), \mu) \rangle + W(x, u(\xi), \mu) - E \right) \frac{|\xi|^2}{2} \\ &+ \frac{|\eta - \hat{b}(x, \xi, \mu)|^2}{2} + \frac{\langle \hat{B}(x, \xi, \mu)(\eta - \hat{b}(x, \xi, \mu)), \eta - \hat{b}(x, \xi, \mu) \rangle}{2|\xi|^2} \\ &+ \frac{1}{2} \langle \hat{C}(x, \xi, \mu)(y - a(x, u(\xi), \mu)), \eta - \hat{b}(x, \xi, \mu) \rangle = \mu \phi(x, u(\xi), \mu). \end{aligned}$$

Solving for μ we obtain the regularized Hamiltonian

$$\mu = \mathcal{H}(z, \xi, \eta) = \mathcal{H}_2(z, \xi, \eta) + O_3(\xi, \eta), \quad z = (x, y), \quad (4.11)$$

where

$$\mathcal{H}_2(z, \xi, \eta) = \frac{(F_0(z) - E)|\xi|^2 + |\eta|^2}{2\phi_0(x)}, \quad \phi_0(x) = \phi(x, 0, 0).$$

Here

$$F_0(z) = H_0(x, y, 0, 0) = \frac{1}{2} \langle A_0(x)(y - a_0(x)), y - a_0(x) \rangle + W_0(x)$$

is the Hamiltonian (2.7) on T^*N corresponding to the Lagrangian $L_0|_{TN}$. Indeed,

$$A_0(x) = A(x, 0, 0), \quad a_0(x) = a(x, 0, 0), \quad W_0(x) = W(x, 0, 0).$$

By (4.9), the regularized Hamiltonian \mathcal{H} is at least of class $C^{3+\text{Lip}}$, and the only source of low regularity is the term $|\xi|^{-2}\hat{B}(x, \xi, \mu) = O_4(\xi)$. In applications to celestial mechanics, \hat{B} is divisible by $|\xi|^2$, so \mathcal{H} is real analytic.

Since in the new symplectic variables x, y, ξ, η the level set $\{H_\mu = E\}$ becomes $\{\mathcal{H} = \mu\}$, the symplectic map

$$\psi(x, y, \xi, \eta) = (x, y, u(\xi), v(\xi, \eta))$$

takes solutions of the regularized Hamiltonian system on the level set $\{\mathcal{H} = \mu\}$ to solutions of the original Hamiltonian system on the level set $\{H_\mu = E\}$ (with different time parametrization).

For fixed z , \mathcal{H}_2 is a quadratic Hamiltonian with eigenvalues

$$\pm \lambda(z), \quad \lambda(z) = \frac{\sqrt{E - F_0(z)}}{\phi_0(x)}, \quad (4.12)$$

each of multiplicity 2. We see that $\xi = \eta = 0$ is a critical manifold for \mathcal{H} and $\mathcal{M} = \mathcal{M}_E = \{(z, 0, 0) : F_0(z) < E\}$ is a normally hyperbolic invariant manifold.

Let $\tilde{U} = D \times B_r$, $r = \sqrt{2\rho}$ and let $\pi(x, \xi) = (x, u(\xi))$. We proved the following semi global version of the Levi-Civita regularization:

Theorem 4.1. *Let $D \Subset N \cap \mathcal{D}_E$ be a domain such that $T^\perp N|_D$ is trivial. There exist a tubular neighborhood U of D , a smooth map $\pi : \tilde{U} \rightarrow U$ and a $C^{3+\text{Lip}}$ Hamiltonian \mathcal{H} on $T^*\tilde{U}$ such that:*

- $\pi : \tilde{U} \setminus \tilde{D} \rightarrow U \setminus D$, $\tilde{D} = \pi^{-1}(D)$, is a double covering branched over D and $\pi : \tilde{D} \rightarrow D$ a diffeomorphism;
- \mathcal{H} is invariant under the sheet interchanging involution $\sigma : \tilde{U} \rightarrow \tilde{U}$;
- π takes trajectories of system $(\tilde{U} \setminus \tilde{D}, \mathcal{H} = \mu)$ to trajectories of system $(U \setminus D, H_\mu = E)$ (with changed time parametrization);
- The Hamiltonian system (\tilde{U}, \mathcal{H}) has a $2(m-2)$ -dimensional normally hyperbolic symplectic critical manifold \mathcal{M} on the level $\mathcal{H} = 0$ with $2(m-2)$ zero eigenvalues and two semisimple real nonzero eigenvalues (4.12), each of multiplicity 2.
- Trajectories of system $(\tilde{U}, \mathcal{H} = 0)$ asymptotic to \mathcal{M}_E are projected by π to trajectories of the degenerate billiard $(M, N, H = E)$ colliding with N .

Since π is a double covering, to each orbit $\gamma : [0, \tau] \rightarrow U$ of the degenerate billiard colliding with N at $x = \gamma(\tau)$, there correspond 2 asymptotic trajectories $\gamma_{1,2} : [0, +\infty) \rightarrow \tilde{U}$, $\gamma_2 = \sigma\gamma_1$, of the regularized system with $\gamma_{1,2}(+\infty) = \pi^{-1}(x)$ and $\{\gamma_1(0), \gamma_2(0)\} = \pi^{-1}(\gamma(0))$. Similarly for an orbit $\gamma : [\tau, 0] \rightarrow U$ with $\gamma(\tau) \in N$.

To prove Theorem 3.2 we use a generalization of the Shilnikov lemma [27] for normally hyperbolic invariant manifolds of a Hamiltonian system. Let \mathcal{M} be a symplectic manifold with symplectic coordinates $z = (x, y)$. Consider a Hamiltonian system with Hamiltonian

$$\mathcal{H}(z, \zeta) = \mathcal{H}_2(z, \zeta) + O_3(\zeta), \quad \mathcal{H}_2 = \frac{1}{2}(a(z)|\eta|^2 - b(z)|\xi|^2), \quad (4.13)$$

where $z \in \mathcal{M}$, $\zeta = (\xi, \eta) \in \mathbb{R}^{2d}$ and $a(z), b(z) > 0$ for $z \in \mathcal{M}$. Thus $(z_0, 0, 0)$ is a hyperbolic equilibrium with nonzero eigenvalues $\pm\lambda(z_0)$, $\lambda(z_0) = \sqrt{a(z_0)b(z_0)}$. Its stable and unstable manifolds are given by

$$W^\pm(z_0) = \{(z, \xi, \eta) : z = g_\pm(z_0, \xi), \eta = h_\pm(z_0, \xi)\}, \quad g_\pm(z_0, 0) = z_0, \quad h_\pm(z_0, 0) = 0.$$

Let $r > 0$. Fix a compact set $\mathcal{K} \subset \mathcal{M}$ and $\varepsilon > 0$ and denote

$$\mathcal{Q}_- = \{(z_0, \xi_-, \xi_+) \in \mathcal{K} \times S_r^2 : \langle \xi_-, \xi_+ \rangle \geq \varepsilon^2 r^2\} \quad (4.14)$$

$$\mathcal{Q}_+ = \{(z_0, \xi_-, \xi_+) \in \mathcal{K} \times S_r^2 : \langle \xi_-, \xi_+ \rangle \leq -\varepsilon^2 r^2\}. \quad (4.15)$$

The next result is a corollary of Theorem 6 in [11].

Theorem 4.2. *There exists $r > 0$ and $\mu_0 > 0$ such that for any*

$$(z_0, \xi_-, \xi_+, \mu) \in \mathcal{X} = (\mathcal{Q}_+ \times (0, \mu_0)) \cup (\mathcal{Q}_- \times (-\mu_0, 0))$$

- *There exists*

$$T \sim -\frac{1}{2\lambda(z_0)} \ln \left(\frac{-\mu}{\langle \xi_+, \xi_- \rangle} \right)$$

and a solution

$$\zeta(t) = (z(t), \xi(t), \eta(t)) \in \mathcal{M} \times B_r \times \mathbb{R}^d, \quad t \in [-T, T],$$

with $\mathcal{H} = \mu$ such that

$$z(0) = z_0, \quad \xi(T) = \xi_-, \quad \xi(-T) = \xi_+. \quad (4.16)$$

- *We have*

$$\xi(0) = \sqrt{\frac{-\mu}{2b(z_0)\langle \xi_+, \xi_- \rangle}} (\xi_+ + \xi_-) + O(\sqrt{|\mu|r}) + O(\mu), \quad (4.17)$$

$$\eta(0) = \sqrt{\frac{-\mu}{2a(z_0)\langle \xi_+, \xi_- \rangle}} (\xi_- - \xi_+) + O(\sqrt{|\mu|r}) + O(\mu), \quad (4.18)$$

$$z_\pm = g_\pm(z_0, \xi_\pm) + O(\mu \ln |\mu|),$$

$$\eta_\pm = h_\pm(z_0, \xi_\pm) + O(\mu).$$

- ζ smoothly depends on $(z_0, \xi_-, \xi_+, \mu) \in \mathcal{X}$ and converges (as a nonparametrized curve), as $\mu \rightarrow 0$, to the concatenation of asymptotic trajectories $\zeta_+ : [0, +\infty) \rightarrow \mathcal{M} \times B_r \times \mathbb{R}^d$ and $\zeta_- : (-\infty, 0] \rightarrow \mathcal{M} \times B_r \times \mathbb{R}^d$ in the stable and unstable manifolds $W^\pm(z_0)$ of $z_0 \in \mathcal{M}$:

$$\zeta_+(0) = (z_+, \xi_+, \eta_+) \in W^+(z_0), \quad \zeta_+(+\infty) = (z_0, 0, 0),$$

$$\zeta_-(0) = (z_-, \xi_-, \eta_-) \in W^-(z_0), \quad \zeta_-(-\infty) = (z_0, 0, 0).$$

- *The Maupertuis action of ζ is a smooth function on \mathcal{X} and has the form*

$$J(\zeta) = \int_\zeta y dx + \eta d\xi = J_-(z_0, \xi_-) + J_+(z_0, \xi_+) + O(\mu \ln |\mu|), \quad (4.19)$$

where $J_\pm(z_0, \xi_\pm)$ are the actions of the asymptotic trajectories ζ_\pm .

- If $\mu < 0$, or $\mu > 0$ and $|\xi_+ - \xi_-| \geq \varepsilon r$, then $|\xi(t)| \geq c\sqrt{|\mu|}$ for $t \in [-T, T]$.

Remark 4.2. In [11] the proof was given for a smooth Hamiltonian \mathcal{H} . This is enough for applications in celestial mechanics. However, one can check that the proof works if $\mathcal{H} \in C^{3+\text{Lip}}$.

Let us prove Theorem 3.2 for $d = 2$. For definiteness let $\mu > 0$. Let $r = \sqrt{2\rho}$. For given $u_{\pm} \in S_{\rho}$ with $u_+ \neq -u_-$ we can find $\xi_{\pm} \in S_r$ such that $u_{\pm} = u(\xi_{\pm})$ and $\langle \xi_+, \xi_- \rangle \neq 0$ by (4.8). Replacing ξ_+ with $-\xi_+$ if necessary (they correspond to the same u_+) we may assume that $\langle \xi_+, \xi_- \rangle > 0$. We conclude that there is $\varepsilon > 0$ such that if $|u_+ + u_-| \geq \delta\rho$, we can find $\xi_{\pm} \in S_r$ with $u_{\pm} = u(\xi_{\pm})$ such that $\langle \xi_-, \xi_+ \rangle \geq \varepsilon r^2$. Then π takes the trajectory in Theorem 4.2 to a trajectory of system $(M \setminus N, H_{\mu} = E)$ satisfying the condition of Proposition 3.1. \square

4.2 Codimension 3

Let $d = 3$. Then Theorem 4.1 is modified as follows:

Theorem 4.3. *There exist an $(m+1)$ -dimensional manifold \tilde{U} , a smooth group action $\Phi_{\theta} : \tilde{U} \rightarrow \tilde{U}$, $\theta \in \mathbb{T}$, a smooth surjective map $\pi : \tilde{U} \rightarrow U$ commuting with Φ_{θ} , and a Φ_{θ} -invariant Hamiltonian $\mathcal{H} \in C^{3+\text{Lip}}$ on $T^*\tilde{U}$ such that:*

- *The group action Φ_{θ} is trivial on $\tilde{D} = \pi^{-1}(D)$ and free on $\tilde{U} \setminus \tilde{D}$. Thus $\pi : \tilde{U} \setminus \tilde{D} \rightarrow U \setminus D$ is a fiber bundle with fiber \mathbb{T} and $\pi : \tilde{D} \rightarrow D$ is a diffeomorphism.*
- *Let G be the momentum integral*

$$G(q, p) = \langle X(q), p \rangle, \quad X(q) = D_{\theta}|_{\theta=0} \Phi_{\theta}(q)$$

of system (\tilde{U}, \mathcal{H}) corresponding to the symmetry group Φ_{θ} . Then π takes trajectories of system $(\tilde{U} \setminus \tilde{D}, \mathcal{H} = \mu)$ with $G = 0$ to trajectories of system $(U, H_{\mu} = E)$.

- *System (\tilde{U}, \mathcal{H}) has a $2(m-2)$ -dimensional normally hyperbolic symplectic critical manifold \mathcal{M} on the level $\{\mathcal{H} = 0, G = 0\}$. Every critical point $z \in \mathcal{M}$ has $2(m-2)$ zero eigenvalues and two semisimple nonzero eigenvalues (4.12), each of multiplicity 4.*
- *Trajectories asymptotic to \mathcal{M} are projected by π to trajectories colliding with N .*

Note that due to symmetry Φ_{θ} to each trajectory $\gamma : [0, \tau] \rightarrow U$ of the billiard colliding with N there correspond a continuum of asymptotic orbits $\tilde{\gamma} : [0, +\infty) \rightarrow \tilde{U}$ of the regularized system with $\tilde{\gamma}(+\infty) = \pi^{-1}(\gamma(\tau))$ and $\tilde{\gamma}(0) \in \pi^{-1}(\gamma(0))$.

Proof. It is similar to the case $d = 2$, only instead of the Levi-Civita regularization we use the KS regularization [23]. The Hamiltonian still has the form (4.6),

but now $u, v \in \mathbb{R}^3$. The square map $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is replaced by the quadratic Hopf map¹⁴ $u : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ given by the Hurwitz matrix $\Gamma(\xi)$:

$$u(\xi) = \frac{1}{2}\Gamma(\xi)\xi, \quad \Gamma(\xi) = \begin{pmatrix} \xi_1 & -\xi_2 & -\xi_3 & \xi_4 \\ \xi_2 & \xi_1 & -\xi_4 & -\xi_3 \\ \xi_3 & \xi_4 & \xi_1 & \xi_2 \end{pmatrix}.$$

It has the following properties:

- $|u(\xi)| = |\xi|^2/2$, $du(\xi) = \Gamma(\xi) d\xi$.

- We have

$$\Gamma(e^{\theta J}\xi) = \Gamma(\xi)e^{\theta J}, \quad J = \begin{pmatrix} 0 & -I_2 \\ I_2 & 0 \end{pmatrix}.$$

Thus $u(e^{\theta J}\xi) = u(\xi)$ is invariant under the group $e^{\theta J} : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ generated by the vector field $J\xi$.

- $u(\xi_+) = u(\xi_-)$ iff $\xi_+ = e^{\theta J}\xi_-$ for some θ .
- $u(\xi_+) = -u(\xi_-)$ iff $|\xi_+| = |\xi_-|$ and $\langle \xi_+, \xi_- \rangle = 0$, $\langle J\xi_+, \xi_- \rangle = 0$. Equivalently, $\langle e^{\theta J}\xi_+, \xi_- \rangle \equiv 0$ for all θ .
- $\Gamma(\xi)\Gamma^*(\xi) = |\xi|^2 I_3$.
- If $\langle J\xi, \eta \rangle = 0$, then $\eta = \Gamma^*(\xi)v$ for a unique $v = v(\xi, \eta) \in \mathbb{R}^3$ given by

$$v = \frac{\Gamma(\xi)\eta}{|\xi|^2}, \quad |\eta| = |\xi||v|, \quad \langle v, du \rangle = \langle \eta, d\xi \rangle.$$

Let

$$Z = \{(\xi, \eta) : \xi \neq 0, \langle J\xi, \eta \rangle = 0\}$$

and let \tilde{Z} be the quotient of Z under the group action $(\xi, \eta) \rightarrow (e^{\theta J}\xi, e^{\theta J}\eta)$. This is a symplectic manifold with a symplectic form derived from $d\eta \wedge d\xi$. The map

$$(\xi, \eta) \in \tilde{Z} \rightarrow (u(\xi), v(\xi, \eta)) \in (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3$$

is invertible and it is a symplectic diffeomorphism:

$$\langle \eta, d\xi \rangle = \langle v(\xi, \eta), du(\xi) \rangle.$$

We make a symplectic change of variables

$$\psi : T^*D \times \tilde{Z} \rightarrow T^*D \times (\mathbb{R}^3 \setminus \{0\}) \times \mathbb{R}^3, \quad \psi(x, y, \xi, \eta) = (x, y, u, v).$$

Define the regularized Hamiltonian $\mathcal{H}(x, y, \xi, \eta)$ on $T^*D \times B_r \times \mathbb{R}^4$ by the same formula (4.11), where \hat{B} and \hat{C} are given by (4.9)–(4.10). It is easy to see that

¹⁴Quaternions provide a simpler formula for u .

the Hamiltonian is invariant under the symplectic transformation $\Phi_\theta(z, \xi, \eta) = (z, e^{\theta J} \xi, e^{\theta J} \eta)$:

$$\mathcal{H}(z, e^{\theta J} \xi, e^{\theta J} \eta) = \mathcal{H}(z, \xi, \eta), \quad z = (\xi, \eta).$$

Indeed,

$$\hat{b}(x, e^{\theta J} \xi, \mu) = e^{\theta J} \hat{b}(x, \xi, \mu), \quad \hat{B}(x, e^{\theta J} \xi, \mu) = e^{\theta J} \hat{B}(x, \xi, \mu) e^{-\theta J}.$$

Hence \mathcal{H} has the Noether integral $G = \langle J\xi, \eta \rangle$. On the zero level set $\{G = 0\}$ we have

$$H_\mu(z, u(\xi), v(\xi, \eta)) = E \quad \Leftrightarrow \quad \mathcal{H}(z, \xi, \eta) = \mu.$$

By a standard result of the Hamiltonian reduction theory (see e.g. [1]), the map $\psi(z, \xi, \eta) = (z, u(\xi), v(\xi, \eta))$ takes trajectories of the regularized system in $\{\mathcal{H} = \mu\} \cap \{G = 0\}$ to trajectories in $\{H_\mu = E\}$. Theorem 4.3 is proved with $\tilde{U} = D \times B_r$, $r = \sqrt{2\rho}$ and $\pi(x, \xi) = (x, u(\xi))$. \square

Let us prove Theorem 3.2 for $d = 3$. For definiteness let $\mu > 0$. Let $r = \sqrt{2\rho}$. For sufficiently small $\varepsilon > 0$ and given $u_\pm \in S_\rho$ with $|u_+ + u_-| \geq \delta\rho$, we need to find $\xi_\pm \in S_r$ with $u_\pm = u(\xi_\pm)$ such that $\langle \xi_-, \xi_+ \rangle \leq -\varepsilon r^2$. Then we can join ξ_+ and ξ_- by a trajectory $\zeta(t) = (z(t), \xi(t), \eta(t))$ in Theorem 4.2. We will show that it is possible to choose ξ_\pm in such a way that this trajectory satisfies $G = \langle J\xi, \eta \rangle \equiv 0$. Then π takes the trajectory in Theorem 4.2 to a trajectory of system $(U \setminus D, H_\mu = E)$ satisfying the conditions of Theorem 3.2.

Let us compute the value of G along the trajectory $\zeta(t)$ in Theorem 4.2. By (4.17)–(4.18),

$$G = \langle J\xi(0), \eta(0) \rangle = -\frac{\mu \langle J\xi_+, \xi_- \rangle}{2\lambda(z_0) \langle \xi_+, \xi_- \rangle} + O(\mu r) = G(z_0, \xi_+, \xi_-, \mu).$$

In the next computation we follow [8]. Suppose that $u_+ + u_- \neq 0$. Then $u_\pm = u(\xi_\pm)$, where $s(\theta) = \langle e^{\theta J} \xi_+, \xi_- \rangle \neq 0$. Let θ_0 be a maximum point of $s(\theta)$. Then

$$s(\theta_0) = \langle e^{\theta_0 J} \xi_+, \xi_- \rangle > 0, \quad s'(\theta_0) = \langle J e^{\theta_0 J} \xi_+, \xi_- \rangle = 0,$$

and the critical point is nondegenerate. By the implicit function theorem for small enough r and μ_0 there is θ near θ_0 such that

$$G(z_0, e^{\theta J} \xi_+, \xi_-, \mu) = 0, \quad \langle e^{\theta J} \xi_+, \xi_- \rangle > 0$$

Then $\tilde{\xi}_+ = e^{\theta J} \xi_+$ satisfies

$$(\tilde{\xi}_+, \xi_-) \in \mathcal{Q}_+, \quad G(z_0, \xi_+, \xi_-, \mu) = 0.$$

Now the trajectory in Theorem 4.2 corresponding to $z_0, \tilde{\xi}_+, \xi_-$ is projected by π to a trajectory of system $(\tilde{U}, H_\mu = E)$ satisfying the conditions of Theorem 3.2. \square

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